Some new results concerning Schur multipliers and duality results between Bergman–Schatten and little Bloch spaces

Liviu-Gabriel Marcoci
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LIVIU-GABRIEL MARCOCI

DEPARTMENT OF MATHEMATICS
LULEÅ UNIVERSITY OF TECHNOLOGY
971 87 LULEÅ, SWEDEN

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Abstract

This Licentiate thesis consists of an introduction and three papers, which deal with some spaces of infinite matrices.

In the introduction we give an overview of the area that serves as a frame for the rest of the thesis.

In Paper 1 we introduce the space $B_w(\ell^2)$ of linear (unbounded) operators on $\ell^2$ which map decreasing sequences from $\ell^2$ into sequences from $\ell^2$ and we find some classes of operators belonging either to $B_w(\ell^2)$ or to the space of all Schur multipliers on $B_w(\ell^2)$.

In Paper 2 we further continue the study of the space $B_w(\ell^p)$ in the range $1 < p < \infty$. In particular, we characterize the upper triangular positive matrices from $B_w(\ell^p)$.

In Paper 3 we prove a new characterization of the Bergman-Schatten spaces $L^p_{\mathcal{B}}(D,\ell^2)$, the space of all upper triangular matrices such that $\|A(\cdot)\|_{L^p(D,\ell^2)} < \infty$, where

$$\|A(r)\|_{L^p(D,\ell^2)} = \left(2 \int_0^1 \|A(r)\|_{C_p}^p r dr \right)^{\frac{1}{p}}.$$ 

This characterization is similar to that for the classical Bergman spaces. We also prove a duality between the little Bloch space and the Bergman-Schatten classes in the case of infinite matrices.
Preface

This Licentiate thesis contains the following papers:

• L. G. Marcoci, L. E. Persson, I. Popa and N. Popa, A new characterization for Bergman-Schatten spaces and a duality result, Research report no. 5, Department of Mathematics, Luleå University of Technology, 2009 (submitted).

These papers are put to a more general frame in an introduction, which also serves as a basic overview of the field.
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Introduction

This Licentiate thesis is dedicated to the study of some Banach spaces of infinite matrices. An important role is played by the upper triangular matrices called analytic matrices as well as some special operators acting on them, for instance Schur multipliers. The first research component of the thesis is strongly related to a classical mathematical object having deep implications in the development of mathematics in the last 150 years: infinite matrices.

The Bloch space has been studied for a long time in complex analysis, for the first time in 1920 by A. Bloch (general references include K. Zhu’s book [45]) regarding the boundary behaviour of normal functions.

There are a few good sources for results and references about Bloch functions on the open unit disk. We mention here the paper of J. M. Anderson, J. Clunie and Ch. Pommerenke [2] and the survey papers of J. M. Anderson [1] and J. Cima [31].

The theory of Bergman spaces has evolved from several sources. A primary model is the related theory of Hardy spaces. For $0 < p < \infty$, a function $f$ analytic in the unit disk $D$ is said to belong to the Hardy space $H^p$ if the integrals $\int_0^{2\pi} |f(re^{i\theta})|^p d\theta$ remain bounded as $r \to 1$. It belongs to the Bergman space $A^p$ if the area integral $\int_D |f(z)|^p d\sigma$ is finite. It is clear that $H^p \subset A^p$.

The structural properties of individual functions in $H^p$ were studied actively in the period 1915-1930, beginning with some classical papers of G. H. Hardy. With the emergence of functional analysis in the 1930’s, $H^p$ spaces began to be viewed as examples of Banach spaces, for $1 \leq p \leq \infty$. This point of view suggested a variety of new problems and provided effective methods for the solution of old problems. We mention here only a few mathematicians who early dealt with this spaces: A. Beurling, H. Shapiro, L. Carleson and A. L. Shields. More generally, we mention here Triebel-Lizorkin spaces, which are very important in the theory of function spaces. Details about these spaces can be found in H. Triebel’s books [42], [43] and [44]. Meanwhile, S. Bergman developed an elegant theory of Hilbert spaces of analytic functions in planar domains and in higher-dimensional complex space, relying heavily on a reproducing kernel that became known as the Bergman
Bergman’s work focused on spaces of analytic functions that are square-integrable over the domain with respect to Lebesgue area or volume measure. When attention was later directed to the spaces $A^p$ over the unit disk, it was natural to call them Bergman spaces.

In the last twenty years the interest concerning these spaces has increased. The pointwise multipliers of the Bloch space and the little Bloch space are characterized by J. Arazy in the case of open disk and by K. Zhu in the case of the open unit ball. Coefficient multipliers of Bloch functions are described by J. M. Anderson and A. L. Shields in their paper. J. Arazy remarked in for the first time a similarity between functions and infinite matrices. This similarity has been the source of many results and conjectures, one of the main result obtained for Schur multipliers is due to G. Bennett in his paper.

**Theorem 1 (Bennett’s Theorem).** The Toeplitz matrix $M$ is a multiplier if and only if there exists a bounded and complex Borel measure $\mu$ on (the circle group) $\mathbb{T}$ with Fourier coefficients $\hat{\mu}(n) = c_n$ for $n = 0, \pm 1, \pm 2, \ldots$.

Moreover, we then have

$$\|M\|_{(2,2)} = \|\mu\| = \|M\|_{(\infty,1)}.$$ 

An infinite matrix $M$ is a Toeplitz matrix if it is on the form

$$m_{jk} = c_{j-k} \ (j, k = 0, 1, 2, \ldots).$$

We call a matrix $M$ a $(p,q)$-multiplier, $1 \leq p, q \leq \infty$, if $M \ast A$ maps $\ell^p$ into $\ell^q$ whenever $A$ does. The set $\mathcal{M}(p,q)$ of all such multipliers becomes a Banach space when it is endowed with the norm

$$\|M\|_{(p,q)} = \sup\{\|M \ast A\|_{p,q} : \|A\| \leq 1\}.$$ 

Another direction of research was that to study vector valued analytic functions, but considered from a Banach space point of view.

In this way appeared a series of papers e.g. by O. Blasco, J. L. Arregui, J. G. Cuerva, A. Pelczyński, Q. H. Xu dedicated to Hardy, Bergman, Bloch and BMO vector valued spaces (see e.g. [14]-[25], [6], [7], [26], [27] and [28]). These papers continue the previous work of D. L. Burkholder and J. Garcia Cuerva and J. L. Rubio de Francia concerning singular operators and vector valued Hardy spaces.

We define the matricial analogue of these spaces according to the corresponding definitions for analytic functions and we use the powerful device of Schur multipliers and its characterization in the case of Toeplitz matrices. Different spaces of infinite matrices, analytic matrices and Schur multipliers can be found in [12], [8], [9], [10], [36], [37], [41], [39] and [40].
One alternative to study the proprieties of some Banach spaces of infinite matrices was to understand better the properties of sequence spaces and to find another characterizations for them. Of course Hardy type inequalities plays an important role here. There is a huge literature in this field but we mention here only the books [34], [35], [11] and [30] and the references given there.

In this thesis we complement the theory and prove some new results in this fascinating field in three papers:

**Paper 1**
In the first paper of this Licentiate thesis we introduce the space of infinite matrices $B_w(\ell^2)$. This Banach space of infinite matrices can be regarded as a weaker version of the space $B(\ell^2)$, the space of all bounded operators from $\ell^2$ into $\ell^2$. It is a weaker version because it consists of those operators which maps the sequences which are decreasing in modulus from $\ell^2$ into $\ell^2$. This space actually appeared in the study of matriceal analogue of classical function spaces like $C(T)$ (the continuous functions on the torus), the Wiener algebra $A(T)$ and the Lebesgue space $L^1(T)$.

It is easy to see that $B(\ell^2) \subset B_w(\ell^2)$ and that the inclusion is proper. It is interesting that these two spaces coincides on the subspace consisting of Toeplitz matrices. More precisely, we prove that the Toeplitz matrix $A$ belongs to $B(\ell^2)$ if and only if $A$ belongs to $B_w(\ell^2)$. Using this result we prove a theorem which is similar to G. Bennett’s Theorem 1 above.

One main theorem of this paper is the following, which is in fact a new characterization of the Toeplitz upper triangular infinite matrices from $B(\ell^2)$:

**Theorem 2.** Let $A = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \ldots \\ 0 & a_0 & a_1 & a_2 & \ldots \\ 0 & 0 & a_0 & a_1 & \ldots \\ 0 & 0 & 0 & a_0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$ be an upper triangular Toeplitz matrix. Then $A \in B(\ell^2)$ if and only if the sublinear operator $T_A$ is bounded from $\ell^2$ into $\ell^2$ where

$$T_A (b)(j) = \frac{1}{j} \sum_{m=0}^{j} \left| (a \ast b)(m) \right|, \quad T_A (b)(0) = \left| a_0 b_0 \right|$$

and $(a \ast b)(m) = \sum_{k+l=m} a_k b_l$, $a = (a_k)_{k \geq 0}$, $b = (b_k)_{k \geq 0} \in \ell^2$. 

Paper 2
In the second paper we deal with the Banach space of infinite matrices $B_w(\ell^p)$, $1 < p < \infty$. In the study of this space an important role is played by the following old result that we obtain with a completely new proof: Let

$$d(q)^{\times} = ces(p)$$

where $d(p) = \{x = \{x_k\}_{k=1}^{\infty} : (\sum_{k=1}^{\infty} \sup_{n \geq k} |x_n|^p)^{\frac{1}{p}} < \infty\}$ and $ces(p) = \{x = \{x_k\}_{k=1}^{\infty} : (\frac{1}{n} \sum_{k=1}^{n} |x_k|)^{p} < \infty\}$.

Here $d(q)^{\times}$ is the associate space of $d(q)$, that is

$$d(q)^{\times} = \{a = (a_n)_n; \text{ such that } \sum_{n=1}^{\infty} |a_n x_n| < \infty \text{ for all } (x_n)_n \in d(q)\}.$$

This result which gives us the Köthe dual of $d(p)$ was obtained also by G. Bennett in [11] by using more technical methods, like factorization of some classical inequalities. This problem was first investigated by A. A. Jagers in 1974 in the paper [33].

In particular, in one important theorem from this paper we characterize some upper triangular operators from $\ell^p$ into another sequence spaces like $d(p)$ and $g(p)$ in terms of some special multipliers:

**Theorem 3.** Let $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ and $B$ be an upper triangular matrix. Then

1. $B \in B(\ell^p, d(p))$ if and only if $B \ast [c] \in B(\ell^p, \ell^1)$ for all $c \in ces(q)$.
2. $B \in B(\ell^p, \ell^p)$ if and only if $B \ast [c] \in B(\ell^p, \ell^1)$ for all $c \in \ell^q$.
3. $B \in B(\ell^q, g(p))$ if and only if $B \ast [c] \in B(\ell^q, \ell^1)$ for all $c \in \ell^q \cdot d(p)$.

Paper 3
In the third paper we deal with analytic matrices. We continue the study of Bergman-Schatten spaces and we give a characterization in terms of Taylor coefficients namely:

**Theorem 4.** Let $A$ be an analytic matrix. Then $A \in L_\sigma^p(D, \ell^2)$ if and only if

$$\sum_{n=0}^{\infty} \frac{1}{(n+1)^2} \|\sigma_n(A)\|^2 < \infty.$$
In the proofs we use very often the powerful tools of Schur multipliers and some techniques from the theory analytic functions theory as well.
Bibliography

BIBLIOGRAPHY


Paper 1
A NEW CLASS OF LINEAR OPERATORS ON \( \ell^2 \) AND SCHUR MULTIPLIERS FOR THEM

ANCA-NICOLETA MARCOCI AND LIVIU-GABRIEL MARCOCI

Abstract. We introduce the space \( B_w(\ell^2) \) of linear (unbounded) operators on \( \ell^2 \) which map decreasing sequences from \( \ell^2 \) into sequences from \( \ell^2 \) and we find some classes of operators belonging either to \( B_w(\ell^2) \) or to the space of all Schur multipliers on \( B_w(\ell^2) \). For instance we show that the space \( B(\ell^2) \) of all bounded operators on \( \ell^2 \) is contained in the space of all Schur multipliers on \( B_w(\ell^2) \).

1. Introduction

Let \( A = (a_{ij})_{i,j \geq 1} \) be an infinite matrix. We define

\[ B_w(\ell^2) = \{ A \text{ infinite matrix : } Ax \in \ell^2 \text{ for every } x \in \ell^2 \text{ with } |x_k| \searrow 0 \}, \]

where

\[ \ell^2 = \left\{ x = (x_k)_k : \sum_{k=0}^{\infty} |x_k|^2 < \infty \right\} \]

is the classical Hilbert space of sequences.

The above space of matrices has appeared in the study of the matriceal analogue of some well-known Banach spaces as \( C(T) \), \( M(T) \), \( L^1(T) \).

A similarity between the functions defined on \( T \) and the infinite matrices was remarked for the first time in 1978 by J. Arazy [1]. Later on, in 1983, A. Shields has exploited further this similarity starting with a few constructs used in harmonic analysis together with their matricial analogues [9].

Recently, in [3], the Fejer’s theory developed for Fourier series was extended in the framework of matrices.

The analogy is as follows: we identify a function \( f \) with the Toeplitz matrix \( A = (a_{ij})_{i,j \geq 1} \),

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\[ a_{ij} = a_{i-j} \text{ for all } i, j \in \mathbb{N}^* \]

where \((a_k)_{k \in \mathbb{Z}}\) is the sequence of Fourier coefficients of the function \(f\).

The Schur product of two matrices is defined by

\[ A \ast B = (a_{ij} \cdot b_{ij})_{i,j \geq 1}, \]

where \(A = (a_{ij})_{i,j \geq 1}, B = (b_{ij})_{i,j \geq 1}\). We denote by

\[ M \left( \ell^2 \right) = \{ M : M \ast A \in B \left( \ell^2 \right) \text{ for every } A \in B \left( \ell^2 \right) \} \]

the space of all Schur multipliers equipped with the following norm

\[ \|M\| = \sup_{\|A\|_{B(\ell^2)} \leq 1} \|M \ast A\|_{B(\ell^2)}. \]

Let us define \(A_k = (a_{ij}')_{i,j \geq 1}, k \in \mathbb{Z}\) to be the matrix with the elements

\[ a_{ij}' = \begin{cases} a_{ij} & \text{if } j - i = k, \\ 0 & \text{otherwise.} \end{cases} \]

\(A_k\) is called the Fourier coefficient of \(k\)-order of the matrix \(A\) (see e.g. [3]).

We have now a similarity between the expansion in Fourier series of a periodical function \(f\) on \(\mathbb{T}\)

\[ f = \sum_k a_k e^{ikt} \]

and the decomposition of matrix in diagonal matrices

\[ A = \sum_{k \in \mathbb{Z}} A_k \]

In this way we can say that \(B(\ell^2)\) represents the matriceal analogue of \(L^\infty(\mathbb{T})\) and \(M(\ell^2)\) is the analogue of \(M(\mathbb{T})\). Moreover in [3] was introduced the space of continuous matrices denoted by \(C(\ell^2)\). \(A\) is a continuous matrix if \(\lim_{n \to \infty} \sigma_n(A) = A\), where the limit is taken in the norm of \(B(\ell^2)\) and \(\sigma_n(A)\) is the Cesaro sum associated to \(S_n(A) := \sum_{k=-n}^{n} A_k\).

This space is a Banach space with the following norm

\[ \|A\|_{C(\ell^2)} = \max_n (\sup \|\sigma_n(A)\|_{B(\ell^2)}, \|A\|_{B(\ell^2)}). \]

In the way described earlier this space represents the matriceal analogue of the space \(C(\mathbb{T})\).
Also N. Popa has defined

\[ A(\ell^2) = \left\{ A \ \text{infinite matrix} : \sup_{l \in \mathbb{Z}^+, k \in \mathbb{Z}} |a^l_k| < \infty \right\} \]

where

\[ A = \begin{pmatrix}
  a^1_0 & a^1_1 & \cdots \\
  a^{-1}_1 & a^2_0 & \cdots \\
  & \vdots & \ddots \\
\end{pmatrix} \]

This was the first attempt to define the matriceal analogue of the Wiener algebra \( A(\mathbb{T}) \) and we may call it matriceal Wiener algebra.

It is easy to see that the subspace of all Toeplitz matrices from \( A(\ell^2) \) can be identified with \( A(\mathbb{T}) \) but since \( A(\ell^2) \not\subseteq C(\ell^2) \) it is necessary to find a larger space than \( B(\ell^2) \). In order to see that \( A(\ell^2) \not\subseteq C(\ell^2) \) let us consider

\[ A = \begin{pmatrix}
  1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
  0 & 0 & 1 & 0 & 0 & 0 & \cdots \\
  0 & 0 & 0 & 1 & 0 & 0 & \cdots \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\
\end{pmatrix} \]

Then \( \sup_{l \in \mathbb{Z}^+, k \in \mathbb{Z}} |a^l_k| = 1 \) and \( A \in A(\ell^2) \) implies \( \|A e_{\frac{n(n+1)}{2}}\|_2 = n \) where \( e_k = (0, \dots, 0, 1, 0, \dots) \).

One solution would be to choose a larger space than \( B(\ell^2) \), and, for this reason we introduce \( B_w(\ell^2) \). Clearly \( B_w(\ell^2) \) is a Banach space with the norm

\[ \|A\|_{B_w(\ell^2)} = \sup_{\|x\|_2 \leq 1, |x_k| > 0} \|Ax\|_2. \]

The following result due to E. Sawyer [8] will be used very often in the sequel. See also [7].

If \( v = (v(n))_n \) is a weight on \( \mathbb{N}^* \), we put \( \tilde{v} = \sum_{n=0}^\infty v(n) \chi_{[n,n+1)} \) and \( \tilde{V}(t) = t \int_0^t \tilde{v}(s) ds. \)

Let us mention moreover that the relation \( f \approx g \) means that there are two positive constants \( a \) and \( b \) such that \( af \leq g \leq bf \).

Then we have:
**Theorem 1.** Let \( w = (w(n))_n, \ v = (v(n))_n \) be weights on \( \mathbb{N}^* \) and let

\[
S = \sup_{f} \left( \sum_{n=0}^{\infty} f(n)w(n) \right)^{\frac{1}{p}}.
\]

Then

(i) If \( 0 < p \leq 1 \)

\[
S = \sup_{n \geq 0} \frac{V(n)}{W^\frac{1}{p}(n)},
\]

with \( W \) defined by \( W(n) = \sum_{k=0}^{n} w(k) \) and similarly for \( V \).

(ii) If \( 1 < p < \infty \),

\[
S \approx \left( \int_{0}^{\infty} \frac{\tilde{V}(t)}{W(t)} \right)^{p'} \frac{1}{p} \approx \left( \int_{0}^{\infty} \frac{\tilde{V}(t)}{W(t)} \right)^{p'} \frac{1}{p} \tilde{w}(t) dt \]

\[
+ \frac{\tilde{V}(\infty)}{W^{\frac{1}{p}}(\infty)} \tilde{w}(t) dt \]

where \( \frac{1}{p} + \frac{1}{p'} = 1 \).

The paper is organized as follows: in Section 2 we show that the matriceal Wiener algebra \( A(\ell^2) \) is a subset of \( B_w(\ell^2) \) (Proposition 2) and we give some criteria for diagonal matrices to belong to \( B_w(\ell^2) \). Moreover we consider the Schur product of matrices and remark that \( B_w(\ell^2) \) is not closed under this product. Using the Sawyer result we prove in Section 3 the main result of the paper, namely that linear and bounded operators on \( \ell^2 \) are Schur multipliers on \( B_w(\ell^2) \), a result which is not obvious, since \( B_w(\ell^2) \) is not Schur algebra. Finally we collected in Section 4 all results concerning the Toeplitz matrices. For instance there is no difference between Toeplitz matrices from \( B(\ell^2) \) and those from \( B_w(\ell^2) \).

**2. Preliminary results**

**Proposition 2.** \( A(\ell^2) \subseteq B_w(\ell^2) \) with \( \|A\|_{B_w(\ell^2)} \leq \|A\|_{A(\ell^2)} \).

**Proof.** Let \( A \in A(\ell^2) \) be an upper triangular matrix. Then for every \( (x_l)_{l=0}^{\infty} \in \ell^2 \) with \( |x_l| \downarrow 0 \) and \( \|x_l\| \leq 1 \) we have

\[
\sum_{k=1}^{\infty} \left| \sum_{l=k-1}^{\infty} a_{l-k+1}^k x_l \right|^2 \leq \sum_{k=1}^{\infty} \left( \sum_{l=k-1}^{\infty} |a_{l-k+1}^k| |x_l| \right)^2 \leq \sum_{k=1}^{\infty} \left( \sum_{l=k-1}^{\infty} |a_{l-k+1}^k| \right)^2 \sup_{l \geq k-1} |x_l|^2 \leq \sum_{k=1}^{\infty} \left( \sum_{l=k-1}^{\infty} |a_{l-k+1}^k| \right)^2 \sup_{l \geq k-1} |x_l|^2 \leq \]
\[ \leq \|A\|_{A(\ell^2)}^2 \sum_{k=1}^{\infty} \sup_{l \geq k-1} |x_l|^2 = \|A\|_{A(\ell^2)}^2 \|x\|_2^2 \leq \|A\|_{A(\ell^2)}^2 \]

which implies that \( \|A\|_{B_w(\ell^2)} \leq \|A\|_{A(\ell^2)}. \) □

**Proposition 3.** Let \( A = A_0 \) given by \((a_k)_{k=1}^{\infty}\). Then \( A \in B_w(\ell^2) \) if and only if

\[
\sup_{n \in \mathbb{N}^*} \left( \frac{n}{n} \sum_{k=1}^{n} |a_k|^2 \right)^{\frac{1}{2}} < \infty.
\]

Moreover

\[
\|A\|_{B_w(\ell^2)} = \sup_{n \in \mathbb{N}^*} \left( \frac{n}{n} \sum_{k=1}^{n} |a_k|^2 \right)^{\frac{1}{2}}.
\]

**Proof.** The sufficiency follows immediately from the factorization

\( \ell^2 = d(1,2) \cdot g(1,2) \)

where

\[
d(1,2) = \left\{ x : \sum_{n=1}^{\infty} \sup_{k \geq n} |x_k|^2 < \infty \right\}
\]

\[
g(1,2) = \left\{ x : \sum_{k=1}^{\infty} |x_k|^2 = O(n) \right\} \text{ see e.g. [5].}
\]

For the necessity take \( x^n = \left( \frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}}, 0, \ldots \right) \), \( \|x^n\|_2 = 1 \), \( |x^n_k| \searrow 0 \), \( n \geq 1 \)

\[
\|Ax^n\|_2 = \left( \frac{1}{n} \sum_{k=1}^{n} |a_k|^2 \right)^{\frac{1}{2}} \leq \|A\|_{B_w(\ell^2)}
\]

and \( \sup_{n} \left( \frac{\sum_{k=1}^{n} |a_k|^2}{n} \right)^{\frac{1}{2}} < \infty. \) □

If we translate this sequence \((a_k)_{k=1}^{\infty}\) above or below of the main diagonal we obtain the following similar result.
Corollary 4. a) Let \( k > 0 \) and \( A = A_k \) given by \((a_k)_{k=1}^{\infty}\). Then \( A \in B_w(\ell^2) \) if and only if
\[
\sup_{n \in \mathbb{N}^*} \left( \frac{1}{n+k} \sum_{l=1}^{n} |a_l|^2 \right)^{\frac{1}{2}} < \infty \text{ and } \\
\|A\|_{B_w(\ell^2)} = \sup_{n \in \mathbb{N}^*} \left( \frac{1}{n+k} \sum_{l=1}^{n} |a_l|^2 \right)^{\frac{1}{2}},
\]
b) Let \( k < 0 \) and \( A = A_k \) given by \((a_k)_{k=1}^{\infty}\). Then \( A \in B_w(\ell^2) \) if and only if
\[
\sup_{n \in \mathbb{N}^*} \left( \frac{1}{n} \sum_{l=1}^{n} |a_l|^2 \right)^{\frac{1}{2}} < \infty \text{ and } \\
\|A\|_{B_w(\ell^2)} = \sup_{n \in \mathbb{N}^*} \left( \frac{1}{n} \sum_{l=1}^{n} |a_l|^2 \right)^{\frac{1}{2}}.
\]

Remark 5. 1. Clearly \( B(\ell^2) \subseteq B_w(\ell^2) \) and using Proposition 3 for the matrix \( A = A_0 \) given by \((a_k)_{k=1}^{\infty}\),
\[
a_k = \begin{cases} 
\sqrt{2^n} & \text{if } k = 2^n \\
0 & \text{if } k \neq 2^n 
\end{cases} \quad n \geq 1
\]
we can easily show that the inclusion is proper.

2. While \( B(\ell^2) \) is closed under Schur multiplication \( B_w(\ell^2) \) is not.
For example it is easy to see that \( A \ast A \notin B_w(\ell^2) \) where \( A \) is the matrix defined previously.

3. The space \( B_w(\ell^2) \) cannot be compared with \( M(\ell^2) \) meaning that
\[
(1) \quad B_w(\ell^2) \nsubseteq M(\ell^2) \text{ and } \\
(2) \quad M(\ell^2) \nsubseteq B_w(\ell^2).
\]

For (1) there is \( A = A_0 \) given by \((a_k)_{k=1}^{\infty}\) such that \((a_n)_n \notin l^\infty\), \( A \in B_w(\ell^2) \) and \( A \notin M(\ell^2) \).

For (2) we take \( A = \begin{pmatrix} 1 & 1 & \ldots & 1 & \ldots \\
0 & 0 & \ldots & 0 & \ldots \\
0 & 0 & \ldots & 0 & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots 
\end{pmatrix} \in M(\ell^2) \) but \( A \notin B_w(\ell^2) \).

4. Let \( A = A_0 \) given by \((a_k)_{k=1}^{\infty}\). Then
\[
A \in M(B_w(\ell^2), B_w(\ell^2)) = \{ M : M \ast B \in B_w(\ell^2) \text{ for every } B \in B_w(\ell^2) \}
\]
if and only if \((a_n)_n \in l^\infty\).
A \in M(B_w(\ell^2), B_w(\ell^2)) then
\[ A \ast B \in B_w(\ell^2) \text{ for } B = B_0 \in B_w(\ell^2) \text{ given by } (b_k)^{\infty}_{k=1}. \]

From Proposition 3,
\[ (b_k)^{\infty}_{k=1} \in g(1, 2), \|Ax\|_{\ell^2}^2 = \sum_{k=1}^{\infty} |a_k b_k x_k|^2 < \infty \]
for every \( b = (b_k)_k \in g(1, 2) \), and \( x \in d(1, 2) \).

Using the factorization \( \ell^\infty \cdot g(1, 2) = g(1, 2) \) and \( \ell^2 = d(1, 2) \cdot g(1, 2) \), provided in [5] we get that \( (a_n)_n \in \ell^\infty \).

Now let \( (a_n)_n \in \ell^\infty \). It is easy to see that
\[ \|B_0\|_{B_w(\ell^2)} \leq \|B\|_{B_w(\ell^2)}. \]

### 3. Main Result

**Lemma 6.**
\[
\sup_{|x_n| \leq 0} \frac{\sum_{n=1}^{\infty} |a_n x_n|^2}{\left( \sum_{n=1}^{\infty} |x_n|^2 \right)^{\frac{1}{2}}} = \sup_{|x_n| \leq 0} \frac{\sum_{n=1}^{\infty} |a_n|^2}{\left( \sum_{n=1}^{\infty} |x_n|^2 \right)^{\frac{1}{2}}} \approx \left( \sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{k=1}^{n} |a_k| \right)^2 \right)^{\frac{1}{2}}
\]

where \((a_n)_n\) and \((x_n)_n\) are sequences of complex numbers.

**Proof.** We denote \( S = \sup_{|x_n| \leq 0} \frac{\sum_{n=1}^{\infty} |a_n x_n|^2}{\left( \sum_{n=1}^{\infty} |x_n|^2 \right)^{\frac{1}{2}}} \). From Theorem 1 we have
\[
S \approx \left( \int_{0}^{\infty} \frac{\tilde{V}(t)}{W(t)} \tilde{w}(t) dt \right)^{\frac{1}{2}} + \frac{\tilde{V}(\infty)}{W^{\frac{1}{2}}(\infty)},
\]
where \( v(n) = |a_n|, w(n) = 1, f(n) = |x_n| \) for every \( n \) nonnegative integer.

In this case
\[
\tilde{v} = \sum_{n=0}^{\infty} v(n) \chi_{[n,n+1)} = \sum_{n=0}^{\infty} |a_n| \chi_{[n,n+1)}(s), \text{ where } a_0 = 0,
\]
therefore, for \( t \in (j, j+1) \), we have
\[
\tilde{V}(t) = \int_{0}^{t} \tilde{v}(s) ds = \int_{0}^{j} \tilde{v}(s) ds + \int_{j}^{t} \tilde{v}(s) ds = \sum_{m=0}^{j-1} \int_{m}^{m+1} \tilde{v}(s) ds + \int_{j}^{t} \tilde{v}(s) ds = \sum_{m=0}^{j-1} |a_m| + |a_j| (t - j),
\]
\[ \tilde{V}(\infty) = \int_0^\infty \tilde{v}(s) \, ds = \int_0^\infty \sum_{n=0}^\infty |a_n| \chi_{[n,n+1)}(s) \, ds = \sum_{n=1}^\infty |a_n| \quad \text{and} \quad \tilde{W}(\infty) = \int_0^\infty \tilde{w}(s) \, ds = \infty, \]

since \( \tilde{w}(s) = \sum_{n=0}^\infty \chi_{[n,n+1)}(s) \).

Clearly, letting \( \tilde{v}_M = \sum_{n=0}^M |a_n| \chi_{[n,n+1)}, \)

\[ \tilde{V}_M = \int_0^\infty \tilde{v}_M(s) \, ds = \int_0^\infty \sum_{n=0}^M |a_n| \chi_{[n,n+1)}(s) \, ds = \sum_{n=1}^M |a_n| < \infty, \]

we get

\[ S = \sup_M \left( \frac{\sum_{n=1}^M |x_n| |a_n|}{\left( \sum_{n=1}^\infty |x_n|^2 \right)^{\frac{1}{2}}} \right) = \sup_M \left[ \int_0^\infty \left( \frac{\tilde{V}_M(t)}{t} \right)^2 \, dt + \frac{\tilde{V}_M(\infty)}{W(\infty)} \right] \approx \left( \int_0^\infty \left( \frac{\tilde{v}(t)}{t} \right)^2 \, dt \right)^{\frac{1}{2}}. \]

But

\[ \int_0^\infty \left( \frac{\tilde{V}(t)}{t} \right)^2 \, dt = \sum_{j=1}^\infty \int_{j-1}^{j+1} \left( \frac{\sum_{m=0}^{j-1} |a_m| + |a_j| (t-j)}{t} \right)^2 \, dt \]

\[ \leq \sum_{j=1}^\infty \left( \int_0^1 \frac{1}{t^2} \, dt \right) \left( \sum_{m=1}^j |a_m| \right)^2 \leq \sum_{j=1}^\infty \left( \frac{1}{j} \sum_{m=1}^j |a_m| \right)^2. \]

On the other hand

\[ \int_0^\infty \left( \frac{\tilde{V}(t)}{t} \right)^2 \, dt \geq \sum_{j=1}^\infty \int_j^{j+1} \left( \frac{\sum_{m=1}^{j-1} |a_m|}{t} \right)^2 \, dt \quad \text{and} \quad \sum_{j=1}^\infty \int_j^{j+1} |a_j|^2 \frac{(t-j)^2}{t^2} \, dt \geq . \]
A NEW CLASS OF LINEAR OPERATORS

\[ \geq \sum_{j=1}^{\infty} \left( \frac{1}{j} \sum_{m=1}^{j} |a_m| \right)^2, \]

which implies that \( S \approx \left( \sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{k=1}^{n} |a_k| \right)^2 \right)^{\frac{1}{2}}. \)

\[ \square \]

**Theorem 7.** \( B(\ell^2) \subseteq M(B_w(\ell^2), B_w(\ell^2)) \).

**Proof.** Let \( A \in B(\ell^2) \) and \( B \in B_w(\ell^2) \). Then

\[ \sum_{j} \left| \sum_{k} a_{jk} b_{jk} x_k \right|^2 \leq \sum_{j} \left( \sum_{k} |a_{jk}| |b_{jk}| |x_k| \right)^2 \leq \sum_{j} \left( \sum_{k} |a_{jk}|^2 \right) \left( \sum_{k} |b_{jk}|^2 |x_k|^2 \right) \]

\[ \leq \sup_{j} \left( \sum_{k} |a_{jk}|^2 \right) \sum_{j} \left( \sum_{k} |b_{jk}|^2 |x_k|^2 \right) \]

and \( \sum_{j} \sum_{k} |b_{jk}|^2 |x_k|^2 = \sum_{k} c_k |x_k|^2 \), where \( c_k = \sum_{j} |b_{jk}|^2 \). Using now Sawyer’s formula for \( p = 1 \) we have

\[ \sup_{|x_k|} \frac{\sum_{k} c_k |x_k|^2}{\sum_{k} |x_k|^2} = \sup_{N} \frac{\sum_{k=1}^{N} c_k}{N} = \sup_{N} \frac{\sum_{k=1}^{N} |b_{jk}|^2}{N} \]

which implies

\[ \| A \ast B \|_{B_w(\ell^2)} \leq \| A \|_{2, \infty} \sup_{N} \left( \frac{\sum_{k=1}^{N} |b_{jk}|^2}{N} \right)^{\frac{1}{2}}. \]

Using Proposition 3 the matrix

\[ C = \begin{pmatrix} c_0^\frac{1}{2} & 0 & 0 & 0 & \cdots \\ 0 & c_1^\frac{1}{2} & 0 & 0 & \cdots \\ 0 & 0 & c_2^\frac{1}{2} & 0 & \cdots \\ 0 & 0 & 0 & c_3^\frac{1}{2} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \in B_w(\ell^2) \]

if and only if \( \sup_{\| x \|_2 \leq 1} \frac{\sum_{k} |b_{jk}|^2}{(\sum_{k} |x_k|^2)^2} < \infty \).
But

\[
\sup_{\|x\|_2 \leq 1} \left( \sum_j |b_{jk}|^2 \right)^{\frac{1}{2}} \leq \sup_{\|x\|_2 \leq 1} \left( \sum_k |x_k|^2 \right)^{\frac{1}{2}} = \sup_j \left( \sum_{|k|=0}^\infty |b_{jk}x_k|^2 \right)^{\frac{1}{2}} = \|B\|_{B_w(\ell^2)} < \infty
\]

which completes the proof. \(\square\)

4. Toeplitz matrices

We give now a necessary condition in order that an upper triangular Toeplitz matrix belong to \(B(\ell^2)\):

**Proposition 8.** Let \(A = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & \ldots \\ 0 & a_0 & a_1 & a_2 & a_3 & \ldots \\ 0 & 0 & a_0 & a_1 & a_2 & \ldots \\ 0 & 0 & 0 & a_0 & a_1 & \ldots \\ 0 & 0 & 0 & 0 & a_0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}\). If \(A \in B(\ell^2)\) then

\[\hat{A} \in B_w(\ell^2), \text{ where } \hat{A} \text{ is the diagonal matrix } \begin{pmatrix} \hat{a}_0 & 0 & 0 & \ldots & \ldots \\ 0 & \hat{a}_1 & 0 & \ldots & \ldots \\ \vdots & \vdots & \ddots & \ddots & \ddots \\ 0 & 0 & \ddots & \ddots & \hat{a}_m \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}\]

is given by \((\hat{a}_m)_{m=0}^\infty\) and \(\hat{a}_m = \sum_{j=0}^m a_j\).

**Proof.** Let \(A \in B(\ell^2)\). Then

\[
\sum_{k=0}^\infty \sum_{j=0}^\infty a_jx_{j+k}^2 < \infty
\]

for every \((x_j)_j \in \ell^2\).

But \(\|e_1 + e_2 + \ldots + e_N\|_2 = N\), where \(e_k = (0, 0, \ldots, 0, 1, 0, \ldots)\).

Thus

\[
A \left( \frac{1}{\sqrt{N}} (e_1 + e_2 + \ldots + e_N) \right) = \frac{1}{\sqrt{N}} (\hat{a}_N, \hat{a}_{N-1}, \ldots, \hat{a}_1, \hat{a}_0, 0, \ldots)
\]
which implies that \( \sup_{N} \frac{1}{N} \sum_{n=0}^{N} |\tilde{a}_{n}|^2 < \infty \) and, by Proposition 3 it follows that \( \tilde{A} \in B_{w}(\ell^2) \).

We remark here that if we translate this result to the case of functions we shall get a necessary condition for a function to belong to \( H^\infty \) i.e. if \( f \in H^\infty \) then

\[
\sup_{N} \frac{1}{N} \left( \sum_{n=0}^{N} |f(n)|^2 \right) < \infty ,
\]

where \( \hat{f}(k) \) is the Fourier coefficient of \( k \)-order.

**Theorem 9.** \( B_{w}(\ell^2) \cap T = B_{w}(\ell^2) \cap T, \) where \( T \) is the set of all Toeplitz matrices.

**Proof.** Let \( A \) be a Toeplitz matrix. Clearly, if \( A \in B_{w}(\ell^2) \) it follows that \( A \in B_{w}(\ell^2) \). It is well known that a Toeplitz matrix \( A = (a_{ij}) \), where \( a_{ij} = a_{i-j} \) for all \( i,j \in \mathbb{N}^* \), maps \( \ell^2 \) into \( \ell^2 \) precisely when there exists a measurable function essentially bounded on \([0,2\pi] \) with Fourier coefficients \( \hat{f}(n) = a_{n} \) \( (n = 0, \pm 1, \pm 2, \ldots) \) and \( \|A\|_{B(\ell^2)} = \|f\|_{\infty} \) see e.g. [11].

Let \( f(t) = \sum_{k=-\infty}^{\infty} a_{k} e^{2\pi i k t} \), \((x_n)_{n=0}^{\infty} \in \ell^2 \) with \(|x_n| \rightarrow 0 \) and \( h(t) = \sum_{k=0}^{\infty} x_{k} e^{2\pi i k t} \).

Then

\[
\|Ax\|_2^2 = \left( \sum_{k=0}^{\infty} \left( \sum_{j=-k}^{k} a_{j} x_{k+j} \right)^2 \right)^{\frac{1}{2}} = \left( \sum_{k=0}^{\infty} \left| \int_{0}^{1} f(t) e^{2\pi i k t} h(-t) \, dt \right|^2 \right)^{\frac{1}{2}} \approx \sup_{\|g\|_2 \leq 1} \left| \sum_{k=0}^{\infty} g_k \int_{0}^{1} f(t) e^{2\pi i k t} h(-t) \, dt \right| = \sup_{\|g\|_2 \leq 1} \left| \int_{0}^{1} f(t) g(t) h(-t) \, dt \right| ,
\]

where \( g(t) = \sum_{k=0}^{\infty} g_k e^{2\pi i k t} \).

Hence

\[
\|A\|_{B_{w}(\ell^2)} = \sup \left\{ \left| \int_{0}^{1} f(t) g(t) h(-t) \, dt \right| : \|g\|_2 \leq 1, \|h\|_2 \leq 1, h \in L^2 ([0,1]) \right\} .
\]

If we take \( g(t) = \frac{1}{\sqrt{\pi}} \sum_{j=0}^{n-1} e^{2\pi i j (t-t_0)} \) and \( h(-t) = \frac{1}{\sqrt{\pi}} \sum_{k=0}^{n-1} e^{2\pi i k (t-t_0)} \) for \( n = 2p+1, \)

\( p \in \mathbb{N}^* \) and \( K_{n} \)-the Fejer’s kernel we obtain

\[
\|A\|_{B_{w}(\ell^2)} \geq \left| \int_{0}^{1} f(t) \frac{1}{2p+1} \left| \sum_{j=-p}^{p} e^{2\pi i j (t-t_0)} \right|^2 \, dt \right| = \left| \int_{0}^{1} f(t) K_{2p+1} (t-t_0) \, dt \right|
\]
which implies that \( \|A\|_{B_w(\ell^2)} \geq \|f\|_\infty = \|A\|_{B(\ell^2)} \).

**Theorem 10.** Let \( A = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \ldots \\ 0 & a_0 & a_1 & a_2 & \ldots \\ 0 & 0 & a_0 & a_1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \) be an upper triangular Toeplitz matrix. Then \( A \in B(\ell^2) \) if and only if the sublinear operator \( T_A \) is bounded from \( \ell^2 \) into \( \ell^2 \) where

\[
T_A (b) (j) = \frac{1}{j} \sum_{m=0}^{j} \left| (a * b) (m) \right|, \quad T_A (b) (0) = |a_0 b_0|
\]

and \( (a * b)(m) = \sum_{k=m}^{\infty} a_k b_l, \ a = (a_k)_{k \geq 0}, \ b = (b_k)_{k \geq 0} \in \ell^2. \)

**Proof.** From Theorem 9, \( A \in B(\ell^2) \) if and only if \( A \in B_w(\ell^2) \) if and only if

\[
\left( \sum_{j=0}^{\infty} a_j x_{j+k} \right)_{k \geq 0} \in \ell^2 \text{ for any } x \in \ell^2 \text{ with } |x_j| \not\subseteq 0. \text{ This is equivalent with}
\]

\[
\sup_{\|b\|_2 \leq 1} \left| \sum_{k=0}^{\infty} \left( \sum_{j=0}^{\infty} a_j x_{j+k} \right) b_k \right| < \infty \ \forall x \in \ell^2, \ |x_j| \not\subseteq 0.
\]

\[
\sum_{k=0}^{\infty} \left( \sum_{j=0}^{\infty} a_j x_{j+k} \right) b_k = \sum_{l=0}^{\infty} x_l c_l, \text{ where } c_l = \sum_{k=0}^{l} a_{l-k} b_k. \text{ From lemma 6}
\]

\[
\sup_{|x_1| \not\subseteq 0} \left( \sum_{l=0}^{\infty} |x_l c_l| \right)^\frac{1}{2} = \sup_{|x_1| \not\subseteq 0} \left( \sum_{l=0}^{\infty} |x_l|^2 \right)^\frac{1}{2} \approx
\]

\[
\approx |a_0 b_0|^2 + \sum_{j=1}^{\infty} \left( \frac{1}{j} \sum_{m=0}^{j} \left| a_{m-k} b_k \right|^2 \right) = |a_0 b_0|^2 + \sum_{j=1}^{\infty} \left( \frac{1}{j} \sum_{m=0}^{j} \left| (a * b) (m) \right| \right)^2.
\]

Thus the proof is complete.

Now we show that \( M \left( B_w(\ell^2), B_w(\ell^2) \right) \cap T \subseteq M(\ell^2) \cap T. \)

**Theorem 11.** If \( M \) is a Toeplitz matrix with \( M = \left( m_{jk} \right), m_{jk} = c_{j-k} \) where \((j,k) = 0, 1, 2, 3, \ldots\) and \( M \in M \left( B_w(\ell^2), B_w(\ell^2) \right) \) there exists \( \mu \) a bounded, complex, Borel measure on \( \mathbb{T} \) with \( \hat{\mu}(n) = c_n \) for \( n = 0, \pm 1, \pm 2, \ldots \). Moreover

\[
\|\mu\| \leq \|M\|_{M(B_w(\ell^2), B_w(\ell^2))}.
\]
Proof. We follow the standard method \[2\] for solving the "problem of moments". Let \( p(t) = \sum_{n=-\infty}^{\infty} p_n e^{int} \) be an arbitrary trigonometric polynomial and denote by \( P \) the Toeplitz matrix generated by \( p \). Then

\[
\left| \sum_{n} c_n p_n \right| = \lim_{N \to \infty} \left| \sum_{n} c_n p_n \left( 1 - \frac{|n|}{N} \right) \right| = \lim_{N \to \infty} \left| \langle (M \ast P) x^{(N)}, x^{(N)} \rangle \right|
\]

where

\[
x^{(N)}(j) = \left\{ \begin{array}{ll}
\sqrt{\frac{1}{N+1}} & \text{for } 0 \leq j \leq N \\
0 & \text{for } j > N
\end{array} \right.
\]

We consider now the linear functional

\[
p \mapsto \sum_{n} c_n p_n
\]

on the subspace of \( C(T) \) generated by polynomials

\[
|\Lambda(p)| = \left| \sum_{n} c_n p_n \right| = \lim_{N \to \infty} \left| \langle (M \ast P) x^{(N)}, x^{(N)} \rangle \right| \leq \left\| (M \ast P) x^{(N)} \right\|_2 \leq
\]

\[
\leq \left\| (M \ast P) \right\|_{\ell^2} \leq \left\| M \right\|_{M(\ell^2, \ell^2)} \cdot \left\| P \right\|_{B_\infty(\ell^2)} = (\text{ by Theorem } 9) = \left\| M \right\|_{M(\ell^2, \ell^2)} \cdot \left\| P \right\|_{\ell^\infty}
\]

which implies that

\[
|\Lambda| \leq \left\| M \right\|_{M(\ell^2, \ell^2)} \cdot \left\| P \right\|_{\ell^\infty}
\]

It is clear now that this map is well-defined and continuous and the existence of a measure satisfying all the requirements of the theorem follows easily from the Hahn-Banach and Riesz representation theorems. \( \square \)

Using an extension of the F. and M. Riesz theorem from \[10\] we have:

**Corollary 12.** If \( M \in M(B_\infty(\ell^2), B_\infty(\ell^2)) \) is a Toeplitz matrix \( M = (m_{jk}) \), \( m_{jk} = c_{j-k} \) \((k, j = 0, 1, 2, \ldots)\), and for \( n < 0 \), \( c_n = 0 \) with \( n \notin E \), where \( E \) is a set of type \( \Lambda(1) \) \((\text{see}[10])\) then \( c_n \to 0 \) as \( n \to \infty \).

**Proof.** This follows immediately from Riemann-Lebesgue lemma. \( \square \)

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References


Department of Mathematics and Informatics, Technical University of Civil Engineering Bucharest, 124 Lacul Tei Boulevard, Bucharest 020396
E-mail address: anca_marcoci@yahoo.com

Department of Mathematics and Informatics, Technical University of Civil Engineering Bucharest, 124 Lacul Tei Boulevard, Bucharest 020396
E-mail address: liviu_marcoci@yahoo.com
Paper 2
SCHUR MULTIPLIERS CHARACTERIZATION OF A
CLASS OF INFINITE MATRICES

A. MARCOCI, L. MARCOCI, L. E. PERSSON, N. POPA

Abstract. Let $B_w(\ell^p)$ denote the space of infinite matrices $A$ for which
$A(x) \in \ell^p$ for all $x = \{x_k\}_{k=1}^{\infty} \in \ell^p$ with $|x_k| \searrow 0$. In this paper
we characterize the upper triangular positive matrices from $B_w(\ell^p)$,
$1 < p < \infty$, by using a special kind of Schur multipliers and the G.
Bennett factorization technique. Also some related results are stated
and discussed.

1. Introduction

In this paper we deal with infinite matrices $A$, whose entries $a^l_k$, for $k \in \mathbb{Z}$
and $l \in \mathbb{Z}^+$, are indexed with respect to the $k$th diagonal and with the $l$th
place on this diagonal. In what follows, sometimes we shall describe an
infinite matrix by $A = (a^l_k)_{k \in \mathbb{Z}, l \in \mathbb{Z}^+}$ more precisely

$$
A = \begin{pmatrix}
   a_1^0 & a_1^1 & a_2^1 & a_3^1 & \cdots \\
   a_0^1 & a_1^2 & a_2^2 & a_3^2 & \cdots \\
   a_{-1}^2 & a_0^3 & a_1^3 & a_2^3 & \cdots \\
   a_{-2}^3 & a_{-1}^4 & a_0^4 & a_1^4 & \cdots \\
   & \cdots & \cdots & \cdots & \cdots \\
\end{pmatrix}
$$

We started our study motivated by the paper [MM], where the first two
authors introduced the space $B_w(\ell^2)$ of those infinite matrices $A$ for which
$A(x) \in \ell^2$ for all $x = \{x_k\}_{k=1}^{\infty} \in \ell^2$ with $|x_k| \searrow 0$.

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principle, Bennett factorization, Wiener algebra and Hardy type inequalities.
This space is of interest because the matrix version of the Wiener algebra $A(T)$, denoted by $A(\ell^2)$, which consists of all infinite matrices $A = (a_{kl})_{k \in \mathbb{Z}, l \in \mathbb{Z}_+}$ such that $\sup_{l \in \mathbb{Z}_+} \sum_{k \in \mathbb{Z}} |a_{kl}| < \infty$, is not contained in the matrix version $C(\ell^2)$ of the space of all continuous functions $C(T)$ (see [BPP] for the definition and the properties of $C(\ell^2)$).

Such an example is given by the matrix

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & \cdots \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
\end{pmatrix},
$$

where on the $\frac{n(n+1)}{2}$-column there are $n$ entries equals to 1 placed on the $\frac{n(n-1)}{2} + 1, \ldots, \frac{n(n+1)}{2}$ rows and 0 otherwise. Clearly we have $\sup_{l \in \mathbb{Z}_+} \sum_{k \in \mathbb{Z}} |a_{kl}| = 1$, hence $A \in A(\ell^2)$ and $J_A e_n = n$ for all $e_n = (0, \ldots, 0, 1, 0, \ldots)$.

It yields that $A(\ell^2) \subset B_w(\ell^2)$ (see Proposition 2 in [MM]), where $B_w(\ell^2)$ is the Banach space with respect to the norm

$$
\|A\|_{B_w(\ell^2)} = \sup_{\|x\|_2 \leq 1, \|x\|_\infty \leq 0} \|A(x)\|_2.
$$

We remark that $\ell^2_{\text{dec}} = \{ x = (x_k) \, \downarrow \, 0, \ x \in \ell^2 \}$ is a cone and the solid hull of this cone, denoted by $so(\ell^2_{\text{dec}}) \subset \ell^2$, coincides with the Banach space $d(2) = \{ x; \sum_{n=1}^{\infty} \sup_{k \geq n} |x_k|^2 < \infty \}$. The spaces $d(p), p \geq 1$ are introduced in [B], where it is described how they are connected to Hardy type inequalities (for historical information and results of this type we refer to the books [KMP] and [KP]). Here $so(\ell^2_{\text{dec}}) = \{ y = (y_k) \in \ell^2 \text{ such that } |y_k| \leq x_k \text{ for all } k \in \mathbb{N}, \text{ where } x_k \downarrow \, 0 \text{ in } \ell^2 \}$.

Let $A$ be a positive matrix, that is such that all the elements of the sequence $A(x)$ are positive whenever $x = (x_j)_j$ is a sequence having only a finite number of nonzero positive elements. Clearly, if $A \in B_w(\ell^2)$, then $A \in B(d(2), \ell^2)$, that is $A$ is a bounded linear operator from $d(2)$ into $\ell^2$. 
The next Lemma, which may be regarded as a discrete version of a special case of the Sawyer duality principle [Sa](see also [KP]) was obtained and applied in [MM].

Lemma 1.1. It yields that

$$\sup_{|x_n|\leq 0} \left| \frac{\sum_{n=1}^{\infty} a_n x_n}{(\sum_{n=1}^{\infty} |x_n|^2)^{1/2}} \right| \approx \left( \frac{1}{n} \sum_{k=1}^{n} |a_k|^2 \right)^{1/2},$$

where \((a_n)_n\) and \((x_n)_n\) are sequences of complex numbers.

For the investigations in this paper we need a corresponding (discrete Sawyer type) result for every \(p > 1\) and not only for \(p = 2\) as in Lemma 1.1 (see our Lemma 2.4).

In this paper we consider the space \(B_w(\ell^p)\), consisting of infinite matrices \(A\) for which \(A(x) \in \ell^p\) for all \(x = \{x_k\}_{k=1}^{\infty} \in \ell^p\) with \(|x_k| \leq 0\) (\(1 < p < \infty\)). In Theorem 2.1 we characterize the upper triangular positive matrices from \(B_w(\ell^p)\) by using a special kind of Schur multipliers. Also some related results are formulated in Section 2. The proofs can be found in Section 3. We pronounce that our proofs are heavily depending on various important factorization results by G. Bennett [B] and Lemma 2.4.

2. Main results

First let us recall the definition of Schur multipliers.

If \(A = (a_{jk})\) and \(B = (b_{jk})\) are matrices of the same size (finite or infinite) their Schur product (or Hadamard product) is defined to be the matrix of elementwise products

$$A * B = (a_{jk} b_{jk}).$$

There is, however, much justification for the term "Schur product" and we refer the reader to [B1] and [St] for an historical discussion. This concept was first investigated by Schur in his paper [S] and has since arisen in several different areas of analysis: [Po], [SS1], [SS2] (complex function theory); [B], [KwP] (Banach spaces); [SW], [P], [BP] (operator theory); [BPP], [BKP] (matricial harmonic analysis) and [St] (multivariate analysis).

If \(X\) and \(Y\) are two Banach spaces of matrices we define Schur multipliers from \(X\) to \(Y\) as the space \(M(X,Y) = \{M : M * A \in Y \text{ for every } A \in X\},\)
equipped with the natural norm

\[ \|M\| = \sup_{\|A\|_X \leq 1} \|MA\|_Y. \]

We use a matrix operation introduced in [BLP], which extends to general matrices, the usual product of a Toeplitz matrix \( A \) and a complex scalar \( c \).

Namely, let \( c = (c^1, c^2, \ldots) \) be a sequence of complex numbers. We denote by \([c]\) the matrix whose entries \([c]_{l,k}\) are equal to \(c^l\), for \(l \geq 1\) and \(k \in \mathbb{Z}\).

We observe that, for a Toeplitz matrix \( A \) and for a constant sequence \( c = (c^1, c^1, \ldots) \), the matrix \([c]A\) coincides with the usual product between the complex number \(c^1\) and the matrix \(A\). Hence we denoted in [BLP] the product \([c]A\) by \(c \odot A\) and considered it as an external product between a matrix and a sequence of complex numbers.

In what follows, using the results about multipliers from [B], we will characterize the upper triangular positive matrices from \(B_w(\ell^p)\) by studying the behaviour of the matrix \([c]\).

Here \(B_w(\ell^p)\) denotes the space of those infinite matrices \(A\) for which \(A(x) \in \ell^p\) for all \(x = \{x_k\}_{k=1}^\infty \in \ell^p\) with \(|x_k| \searrow 0\). It is clear that for \(p > 1\) this is a Banach space with respect the norm

\[ \|A\|_{B_w(\ell^p)} = \sup_{\|x\|_p \leq 1, \ |x_k| \searrow 0} \|A(x)\|_p. \]

Here, as usual,

\[ \ell^p = \{x = \{x_k\}_{k=1}^\infty : \left( \sum_{k=1}^\infty |x_k|^p \right)^{1/p} < \infty\}. \]

Moreover, let

\[ d(p) = \{x = \{x_k\}_{k=1}^\infty : \left( \sum_{k=1}^\infty \sup_{n \geq k} |x_n|^p \right)^{1/p} < \infty\}. \]

Our first result reads:

**Theorem 2.1.** Let \(B\) be an upper triangular matrix. Then

\[ B \in B(\ell^p, ces(p)), \quad 1 < p < \infty, \]

if and only if

\[ B \circ [c] \in B(\ell^p, \ell^1), \quad \text{for all } c \in d(p). \]
Here
\[ \text{ces}(p) = \{ x = \{ x_k \}_{k=1}^\infty \, \text{with} \, \sum_{n=1}^\infty \left( \frac{1}{n} \sum_{k=1}^n |x_k| \right)^p < \infty \} \]
denotes the Banach space equipped with the norm
\[ \| x \|_{\text{ces}(p)} = \left( \sum_{n=1}^\infty \left( \frac{1}{n} \sum_{k=1}^n |x_k| \right)^p \right)^{1/p}. \]

Now we can state our main result concerning the characterization of the matrices belonging to \( B_w(\ell^p) \).

**Theorem 2.2.** A lower triangular positive matrix \( A \) belongs to \( B_w(\ell^p) \), \( 1 < p < \infty \), if and only if \( A^* \circ [c] \in B(\ell^q, \ell^1) \), where \( \frac{1}{p} + \frac{1}{q} = 1 \) for all \( c \in d(p) \), where \( A^* \) is the usual adjoint of the matrix \( A \).

Besides the \( \text{ces}(p) \)-spaces who have already attracted a fair deal of attention in the literature, an important role is played by \( \ell^p, d(p) \) and also \( g(p) \), defined by
\[ g(p) = \{ x = \{ x_k \}_{k=1}^\infty : \sup_{n \geq 1} \left( \frac{1}{n} \sum_{k=1}^n |x_k| \right)^{\frac{1}{p}} < \infty \}. \]

Therefore we also state the following result where \( \text{ces}(p) \) in Theorem 2.2 is replaced by any of these spaces and \( \ell^q \cdot d(p) \) is the sequence space of coordinatewise products (see [B] for further details).

**Theorem 2.3.** Let \( 1 < p < \infty \), \( \frac{1}{p} + \frac{1}{q} = 1 \) and \( B \) be an upper triangular matrix. Then

1. \( B \in B(\ell^p, d(p)) \) if and only if \( B \circ [c] \in B(\ell^p, \ell^1) \) for all \( c \in \text{ces}(q) \).
2. \( B \in B(\ell^p, \ell^p) \) if and only if \( B \circ [c] \in B(\ell^p, \ell^1) \) for all \( c \in \ell^q \).
3. \( B \in B(\ell^p, g(p)) \) if and only if \( B \circ [c] \in B(\ell^p, \ell^1) \) for all \( c \in \ell^q \cdot d(p) \).

Our proof of Theorem 2.1 (and thus of Theorem 2.2) is heavily depending on the following extension of Lemma 1.1 of independent interest.
Lemma 2.4. If $p > 1$, then

$$\sup_{|x_n| \neq 0} \frac{\sum_{n=1}^{\infty} |a_n x_n|}{\left( \sum_{n=1}^{\infty} |x_n|^p \right)^{\frac{1}{p}}} = \sup_{|x_n| \neq 0} \frac{\sum_{n=1}^{\infty} |a_n x_n|}{\left( \sum_{n=1}^{\infty} |x_n|^p \right)^{\frac{1}{p}}} \approx \left( \sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{k=1}^{n} |a_k| \right)^{\frac{q}{q}} \right)^{\frac{1}{q}},$$

where $(a_n)_n$ and $(x_n)_n$ are sequences of complex numbers and $\frac{1}{p} + \frac{1}{q} = 1$.

Remark 2.5. If $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, then

$$d(q)^{\times} = ces(p).$$

Here $d(q)^{\times}$ is the associate space of $d(q)$, that is

$$d(q)^{\times} = \{ a = (a_n)_n; \text{ such that } \sum_{n=1}^{\infty} |a_n x_n| < \infty \text{ for all } (x_n)_n \in d(q) \}.$$

This result which gives us the Köthe dual of $d(p)$ has been obtained also by G. Bennett in [B] using more technical methods, like factorization of some classical inequalities. This problem was first investigated by Jagers in 1974 in the paper [J].

Finally we note that

$$so(\ell^p_{\text{dec}}) = d(p)$$

($\ell^p_{\text{dec}}$ denotes the subspace of $\ell^p$ consisting of non-increasing sequences) and, hence, our results in particular implies Corollary 12.17 in paper [B] by G. Bennett.

3. Proofs

We first present a proof of the crucial Lemma 2.4, which is based on the following result of E. Sawyer [Sa]. For $p \leq 1$ a similar result has been proved by M. J. Carro and J. Soria in their paper [CS].

Lemma 3.1. Let $w = \{w(n)\}_{n=1}^{\infty}$, $v = \{v(n)\}_{n=1}^{\infty}$ be weights on $\mathbb{N}^*$, let

$$S = \sup_{f} \frac{\sum_{n=0}^{\infty} f(n)w(n)}{\left( \sum_{n=0}^{\infty} f(n)^p w(n) \right)^{\frac{1}{p}}}$$
and \( \tilde{v} = \sum_{n=0}^{\infty} v(n) \chi_{[n,n+1)} \), \( \tilde{w} = \sum_{n=0}^{\infty} w(n) \chi_{[n,n+1)} \) and \( \tilde{V}(t) = \int_0^t \tilde{v}(s)ds \),
\( \tilde{W}(t) = \int_0^t \tilde{w}(s)ds \).

If \( 1 < p < \infty \), then
\[
S \approx \left( \int_0^\infty \left( \frac{\tilde{V}(t)}{\tilde{W}(t)} \right)^{\frac{q-1}{q}} \tilde{w}(t)dt \right)^{\frac{1}{q}} \approx \left( \int_0^\infty \left( \frac{\tilde{V}(t)}{\tilde{W}(t)} \right)^{\frac{q}{p}} \tilde{w}(t)dt \right)^{\frac{1}{q}} + \frac{\tilde{V}(\infty)}{\tilde{W}(\infty)}^{\frac{1}{p}}\frac{1}{q}
\]
where \( \frac{1}{p} + \frac{1}{q} = 1 \).

Here, as usual, the relation \( f \approx g \) means that there are two positive constants \( C_0 \) and \( C_1 \) so that \( C_0 f(t) \leq g(t) \leq C_1 f(t) \), \( t \in [0, \infty) \).

**Proof of Lemma 2.4.** We denote
\[
S = \sup_{|x_n| \to 0} \left( \sum_{n=1}^{\infty} |a_n||x_n| \right)^{\frac{1}{p}}.
\]

According to Lemma 3.1 we have that
\[
S \approx \left( \int_0^\infty \left( \frac{\tilde{V}(t)}{\tilde{W}(t)} \right)^{\frac{q}{p}} \tilde{w}(t)dt \right)^{\frac{1}{q}} + \frac{\tilde{V}(\infty)}{\tilde{W}(\infty)}^{\frac{1}{p}}\frac{1}{q},
\]
where \( v(n) = |a_n|, w(n) = 1, f(n) = |x_n| \) for every nonnegative integer \( n \).

In this case
\[
\tilde{v} = \sum_{n=0}^{\infty} v(n) \chi_{[n,n+1)} = \sum_{n=0}^{\infty} |a_n| \chi_{[n,n+1)}, \text{ where } a_0 = 0.
\]

Therefore, for \( t \in (j, j+1) \), it yields that
\[
\tilde{V}(t) = \int_0^t \tilde{v}(s)ds = \int_0^j \tilde{v}(s)ds + \int_j^t \tilde{v}(s)ds
\]
\[
= \sum_{m=0}^{j-1} \int_m^{m+1} \tilde{v}(s)ds + \int_j^t \tilde{v}(s)ds = \sum_{m=0}^{j-1} |a_m| + |a_j|(t-j),
\]
\[
\tilde{V}(\infty) = \int_0^\infty \tilde{v}(s)ds = \sum_{n=0}^{\infty} \int_0^{\infty} |a_n| \chi_{[n,n+1)}(s)ds = \sum_{n=1}^{\infty} |a_n|.
\]
and
\[ \tilde{W}(\infty) = \int_0^\infty \tilde{w}(s)ds = \infty, \]
since \( \tilde{w}(s) = \sum_{n=0}^{\infty} \chi_{[n,n+1)}(s) \).

Letting \( \tilde{v}_M = \sum_{n=0}^{M} |a_n| \chi_{[n,n+1)} \), \( \tilde{V}_M = \int_0^{\infty} \tilde{v}_M(s)ds = \int_0^{\infty} \sum_{n=0}^{M} |a_n| \chi_{[n,n+1)}(s)ds = \sum_{n=1}^{M} |a_n| < \infty \), we get that
\[ S = \sup_{|x_n| \neq 0} \left( \sum_{n=1}^{\infty} |a_n||x_n| \right)^{\frac{1}{q}} \approx \sum_{n=1}^{\infty} \frac{|a_n||x_n|}{\left( \sum_{n=1}^{\infty} |x_n|^p \right)^{\frac{1}{p}}} \approx \sum_{n=1}^{\infty} \left( \sum_{m=0}^{j-1} |a_m| + |a_j|(t-j) \right)^{q} \]
\[ \approx \sum_{j=1}^{\infty} \left( \frac{1}{j} \sum_{m=1}^{j} |a_m| \right)^{q}, \]
which implies that
\[ S \approx \left( \sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{k=1}^{n} |a_k| \right)^{q} \right)^{\frac{1}{q}}. \]
The proof is complete.

**Proof of Theorem 2.1.** For clearness we first prove the theorem for the special case \( p = q = 2 \). Note that \( A^* \) is an upper triangular matrix.
Let $C \overset{\text{def}}{=} \{c\}$ be the upper triangular matrix obtained from $[c]$ taking the triangular projection $P_T$, which acts as follows:

$$P_T(A) = \begin{cases} 
  a_{ij} & \text{if } i \leq j \\
  0 & \text{otherwise.}
\end{cases}$$

(See [BLP].)

Let $B$ be an upper triangular matrix from $B(\ell^2, \text{ces}(2))$. We have $B(x) = \left(\sum_{j=1}^{\infty} b_{ij} x_j\right)_{i=1}^{\infty} \in \text{ces}(2)$, for all $x = (x_j)_{j=1}^{\infty} \in \ell^2$. But $(B \ast C)(x) = \left(\sum_{j=1}^{\infty} b_{ij} c_j x_j\right)^\infty_{i=1}$ is the product of two sequences, one from $\text{ces}(2)$, and the other one completely arbitrary. By Proposition 15.4 in [B] we have that $d(2) = I(2, 2) \overset{\text{def}}{=} \{m : \sum_{k=1}^{\infty} |i_k - i_{k-1}| m_{ik} |^2 < \infty; \text{for each sequence } i \text{ of integers with } i_0 = 0 < i_1 < i_2 < \ldots\}$.

Then, by using the table 29 on page 70 in [B], we get that $(B \ast C)(x) \in \ell^1$, where $c \in d(2) = I(2, 2)$ and $x \in \ell^2$. Hence $B \ast C \in B(\ell^2, \ell^1)$.

Conversely, let $B \ast C \in B(\ell^2, \ell^1)$ for each $c \in d(2)$. By Hölder’s inequality we have that $\ell^1 = \ell^2 \cdot \ell^2$, and, in view of Theorem 3.8 in [B], it follows that $\ell^2 = g(2) \cdot d(2)$, where

$$g(2) = \left\{ x; \sup_n \frac{\sum_{k=1}^{n} |x_k|^2}{n} < \infty \right\}.$$  

Hence $\ell^1 = (\ell^2 \cdot g(2)) \cdot d(2)$ and, according to Theorem 4.5 in [B], it yields that $\ell^1 = \text{ces}(2) \cdot d(2)$. On the other hand, by Proposition 14.5 in [B] $\text{ces}(2)$ has $d(2)$-cancellation property, that is the inclusion $y \cdot d(2) \subset \text{ces}(2) \cdot d(2)$ implies that $y \in \text{ces}(2)$.

Now, by hypotheses, for each $x \in \ell^2$, we have that

$$(B \ast C)(x) = \left(\sum_{j=1}^{\infty} b_{ij} c_j x_j\right)_{i=1}^{\infty} \in \ell^1 = \text{ces}(2) \cdot d(2),$$

for all $c \in d(2)$. By the cancellation property it follows that

$$B(x) = \left(\sum_{j=1}^{\infty} b_{ij} x_j\right)_{i=1}^{\infty} \in \text{ces}(2),$$

that is, by the closed graph theorem, $B \in B(\ell^2, \text{ces}(2))$.

Now we consider the case $p \neq 2$. 
If $B \in B(\ell^p, \text{ces}(p))$, $c \in d(q)$, $q > 1$, and $\frac{1}{p} + \frac{1}{q} = 1$, then as in the proof of the case $p = q = 2$ we have that $d(q) = I(q, q)$ and, in view of the table on page 70 in [B], it follows that

$$(B \ast C)(x) \in \ell^1, \text{ for all } x \in \ell^p,$$

that is $B \ast C \in B(\ell^p, \ell^1)$.

Conversely, let $B \ast C \in B(\ell^p, \ell^1)$ for all $c \in d(q)$. Then, similarly as in the proof of the case $p = q = 2$ we find that

$$\ell^1 = \ell^p \cdot \ell^q = \ell^p \cdot g(q) \cdot d(q) = (\text{by Theorem 4.5 in [B]}) = \text{ces}(p) \cdot d(q).$$

Since $\text{ces}(p)$ has $d(q)$-cancellation property (see Proposition 14.5 in [B] ) it follows that $B \in B(\ell^p, \text{ces}(p))$.

The proof is complete. □

Proof of Theorem 2.2. First let us note that by using Lemma 2.4 it follows that $A \in B_w(\ell^p)$ if and only if $A^* \in B(\ell^q, \text{ces}(q))$, for $\frac{1}{p} + \frac{1}{q} = 1$. It remains to apply Theorem 2.1. □

Proof of Theorem 2.3. (1). If $B \in B(\ell^p, d(p))$, $c \in \text{ces}(q)$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $x \in \ell^p$ we have that

$$(B \ast [c])(x) = \left(\sum_{j=1}^{\infty} b_{ij} x_j \right)^c = \left(\sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} b_{ij} x_j \right) c^j \right)^i \in d(p) \cdot \text{ces}(q) = \left(\sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} b_{ij} x_j \right) c^j \right)^i \in d(p) \cdot \text{ces}(q) =$$

(by Corollary 12.17 in [B]) = $d(p) \cdot d(p)^* \subset \ell^1$.

Hence, by the closed graph theorem, it yields that

$B \ast [c] \in B(\ell^p, \ell^1)$.

Conversely, if $B \ast [c] \in B(\ell^p, \ell^1)$ for all $c \in \text{ces}(q)$, then, denoting by $(y_i)_{i \in \mathbb{N}} = \left(\sum_{j=1}^{\infty} b_{ij} x_j \right)_{i \in \mathbb{N}}$, we have that $(y_i c^j)_{i \in \mathbb{N}} \in \ell^1$ for all $c \in \text{ces}(q)$. Thus

$$(y_i)_{i \in \mathbb{N}} \in \text{ces}(q)^* = (\text{by Corollary 12.17 in [B]}) = d(p),$$

that is

$B \in B(\ell^p, d(p))$.

(2). If $B \in B(\ell^p, \ell^p)$, $x \in \ell^p$, $c \in \ell^q$ with $\frac{1}{p} + \frac{1}{q} = 1$, then, by Hölder’s inequality, $B \ast [c] \in B(\ell^p, \ell^1)$. 

Conversely, let \((y_i c_i)_i \in \ell^1\) for all \(c_i \in \ell^q\). Then \((y_i)_i \in \ell^p\) and, consequently,

\[
B \in B(\ell^p, \ell^p).
\]

(3). If \(B \in B(\ell^p, g(p))\) and \(c \in \ell^q \cdot d(p)\), then, by using the previous notations, we find that

\[
(y_i c_i)_i \in g(p) \cdot \ell^q \cdot d(p) = \ell^p \cdot \ell^q \subset \ell^1.
\]

Conversely, let \((y_i)_i \cdot \ell^q \cdot d(p) \in \ell^1\) (by Theorem 3.8 in [B]) = \(g(p) \cdot d(p) \cdot \ell^q\). Consequently we have to show that

\[
(y_i)_i \in g(p).
\]

This fact follows clearly if \(g(p)\) has the \(d(p) \cdot \ell^q\)-cancellation property.

We note that by using Proposition 14.5 in [B], we get that \((y_i)_i \cdot d(p) \in g(p) \cdot d(p) \cdot \ell^p\).

Indeed, let \((z_i)_i \in d(p)\) be fixed. Then \((y_i z_i)_i \cdot \ell^q \in \ell^p \cdot \ell^q\). Since \(\ell^p\) has the \(\ell^q\)-cancellation property (see Proposition 14.5 in [B]) it follows that

\[
(y_i z_i)_i \in \ell^p = g(p) \cdot d(p), \text{ for all } (z_i)_i \in d(p);
\]

in other words \((y_i)_i \cdot d(p) \in g(p) \cdot d(p)\). Using now the fact that \(g(p)\) has \(d(p)\)-cancellation property, it follows that \((y_i)_i \in g(p)\). The proof is complete. 

\[\square\]

References


[Po] Chr. Pommerenke, Univalent Functions, Hubert, Gottingen 1975. Zbl 0298.30014


Department of Mathematics and Informatics, Technical University of Civil Engineering Bucharest, 124 Lacul Tei Boulevard, Bucharest 020396, ROMANIA E-mail: anca_marcoci@yahoo.com E-mail: liviu_marcoci@yahoo.com

Department of Mathematics, Luleå University of Technology, SE-97187 Luleå, SWEDEN E-mail: larserik@sm.luth.se

Department of Mathematics, University of Bucharest and Institute of Mathematics of Romanian Academy, P.O. BOX 1-764 RO-014700 Bucharest, ROMANIA E-mail: npopa@imar.ro
Paper 3
A NEW CHARACTERIZATION OF BERGMAN-SCHATTEN SPACES AND A DUALITY RESULT

LIVIU-GABRIEL MARCOCI, LARS-ERIK PERSSON, IRINA POPA, AND NICOLAE POPA

DEPARTMENT OF MATHEMATICS AND INFORMATICS
TECHNICAL UNIVERSITY OF CIVIL ENGINEERING BUCHAREST
RO-020396 BUCHAREST, ROMANIA
AND
DEPARTMENT OF MATHEMATICS
LULEÄ UNIVERSITY OF TECHNOLOGY
SE-97 187 LULEÄ, SWEDEN
E-mail: liviu.marcoci@ltu.se

DEPARTMENT OF MATHEMATICS
LULEÄ UNIVERSITY OF TECHNOLOGY
SE-97 187 LULEÄ, SWEDEN
E-mail: larserik@sm.luth.se

DEPARTMENT OF MATHEMATICS AND INFORMATICS
TECHNICAL UNIVERSITY OF CIVIL ENGINEERING BUCHAREST
RO-020396 BUCHAREST, ROMANIA
E-mail: irina.popa09@gmail.com

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF BUCHAREST
AND INSTITUTE OF MATHEMATICS OF ROMANIAN ACADEMY
P.O. BOX 1-764 RO-014700 BUCHAREST, ROMANIA
E-mail: npopa@imar.ro

**Abstract.** Let $B_0(D,ℓ^2)$ denote the space of all upper triangular matrices $A$ such that $\lim_{r \to 1^-} (1 - r^2) \|(A * C(r))'\|_{B(ℓ^2)} = 0$. We also denote by $B_{0,c}(D,ℓ^2)$ the closed Banach subspace of $B_0(D,ℓ^2)$ consisting of all upper triangular matrices whose diagonals are compact operators.

In this paper we give a duality result between $B_{0,c}(D,ℓ^2)$ and the Bergman-Schatten spaces $L^1_a(D,ℓ^2)$. We also give a characterization of the more general Bergman-Schatten spaces $L^p_a(D,ℓ^2), 1 \leq p < \infty$, in terms of Taylor coefficients, which is similar to that of M. Mateljevic and M. Pavlovic [12] for classical Bergman spaces.

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**Department of Mathematics**  
**Luleå University of Technology**  
**SE-971 87 Luleå, Sweden**
1. Introduction

The Bloch and Bergman spaces have been studied for a long time in complex analysis and in the last twenty years the interest concerning these spaces has increased. A direction of research was that to study vector valued analytic functions, but considered from a Banach point of view. In this way appeared a series of papers e.g. by J. A. Arregui, O. Blasco [2] and [3] and O. Blasco [6]–[9]. In what follows we consider the Bloch and Bergman spaces in the framework of matrices e.g. infinite matrix valued functions. We use the powerful device of Schur multipliers and its characterizations in the case of Toeplitz matrices to prove the main theorems. The extension to the matriceal framework is based on the fact that there is a natural correspondence between Toeplitz matrices and formal Fourier series associated to 2π-periodic functions (see e.g [1], [4], [11] and [14]).

Let $A = (a_{jk})$ and $B = (b_{jk})$ be matrices of the same size (finite or infinite). Then their Schur product (or Hadamard product) is defined to be the matrix of elementwise products:

$$A \ast B = (a_{jk} b_{jk}).$$

If $X$ and $Y$ are two Banach spaces of matrices we define Schur multipliers from $X$ to $Y$ as the space $\mathcal{M}(X,Y) = \{M : M \ast A \in Y \text{ for every } A \in X\}$, equipped with the natural norm

$$\|M\| = \sup_{\|A\|_{X} \leq 1} \|M \ast A\|_{Y}.$$ 

In the case $X = Y = B(\ell^{2})$, where $B(\ell^{2})$ is the space of all linear and bounded operators on $\ell^{2}$, the space $M(B(\ell^{2}), B(\ell^{2}))$ will be denoted $M(\ell^{2})$ and a matrix $A \in M(\ell^{2})$ will be called Schur multiplier. We mention here an important result due to G. Bennett [5], which will be often used in this paper.

**Theorem 1.1.** The Toeplitz matrix $M = (c_{j-k})_{j,k}$, where $(c_{n})_{n \in \mathbb{Z}}$ is a sequence of complex numbers, is a Schur multiplier if and only if there exists a bounded and complex Borel measure $\mu$ on (the circle group) $\mathbb{T}$ with

$$\hat{\mu}(n) = c_{n}, \text{ for } n = 0, \pm 1, \pm 2, \ldots.$$

Moreover, we then have that

$$\|M\| = \|\mu\|.$$ 

We will denote by $A_{k}$, the $k^{th}$ diagonal matrix associated to $A$ (see [4]). For an infinite matrix $A = (a_{ij})$ and an integer $k$ we denote by $A_{k}$ the matrix whose...
the preceding sum is a formal one and associated to the Cauchy kernel.

Let $A = (a_{ij})_{i,j \geq 0}$ be an infinite matrix with complex entries and let $n \geq 0$. The matrix $A$ is said to be of $n$-band type (see [4]) if $a_{ij} = 0$ for $|i - j| > n$.

In what follows we will recall some definitions from [13], which we will use in this paper. We consider on the interval $[0, 1)$ the Lebesgue measurable infinite matrices with $A$ we identify the analytic matrices $A$ matrix-valued functions defined on the unit disc $D$ using the correspondence $A(r) \to f_A(r,t) = \sum_{k=-\infty}^{\infty} A_k(r) e^{ikt}$, where $A_k(r)$ is the $k$th-diagonal of the matrix $A(r)$, the preceding sum is a formal one and $t$ belongs to the torus $\mathbb{T}$. This matrix $A(r)$ is called analytic matrix if there exists an upper triangular infinite matrix $A$ such that, for all $r \in [0,1)$, we have $A_k(r) = A_{kr}^k$, for all $k \in \mathbb{Z}$. In what follows we identify the analytic matrices $A(r)$ with their corresponding upper triangular matrices $A$ and we call them also analytic matrices.

We will denote by $C_p$, $0 < p < \infty$, the Schatten class operators (see e.g. [16]). We also recall the definition of Bergman-Schatten spaces (see e.g. [13]). Let $1 \leq p < \infty$. We denote

$$L^p(D, \ell^2) = \{ r \to A(r) \text{ which are strong measurable }C_p- \text{valued functions defined on } [0, 1) \text{ such that} \}$$

$$\|A\|_{L^p(D, \ell^2)} := \left( \int_0^1 \| A(r) \|_{C_p, r}^p \, dr \right)^{\frac{1}{p}} < \infty.$$

Let $\tilde{L}^p_\ell(D, \ell^2) = \{ r \to A(r), \text{ where } A(r) = A \ast C(r) \text{ and } A \text{ are upper triangular matrices with } \| A \|_{L^p(D, \ell^2)} < \infty \}$. Here $C(r)$ denotes the Toeplitz matrix associated to the Cauchy kernel $\frac{1}{1-r}$, for $0 \leq r < 1$. By $\tilde{L}^p_\ell(D, \ell^2)$ we mean the space of all upper triangular matrices such that $\| A \|_{L^p(D, \ell^2)} < \infty$. We identify $\tilde{L}^p_\ell(D, \ell^2)$ and $L^p_\ell(D, \ell^2)$ and call $L^p_\ell(D, \ell^2)$ Bergman-Schatten classes. For $p = \infty$ we denote $L^\infty(D, \ell^2) = \{ r \to A(r) \text{ being a } w^* - \text{measurable function on } [0, 1) : \} \|A\|_{L^\infty(D, \ell^2)} := \text{ess sup}_{0 \leq r < 1} \| A(r) \|_{B(\ell^2)} < \infty \}$ and $L^\infty(D, \ell^2)$ is the subspace of $L^\infty(D, \ell^2)$ consisting of all strong measurable functions on $[0, 1)$. We also consider

$$L^\infty_{an}(D, \ell^2) := \{ A \text{ analytic matrix } : \| A \|_{L^\infty(D, \ell^2)} := \sup_{0 \leq r < 1} \| C(r) \ast A \|_{B(\ell^2)} = \| A \|_{L^\infty(D, \ell^2)} < \infty \}.$$

An important tool in this paper is the Bergman projection. It is known (see e.g. [13]) that for all functions $A(r) \in L^2(D, \ell^2)$ defined on $[0, 1)$ and for all $i, j \in \mathbb{N}$ we
have that
\[ P(A)(r)(i, j) = \begin{cases} 2(j - i + 1)r^{j-i} \int_0^1 a_{ij}(s) \cdot s^{j-i+1}ds, & \text{if } i \leq j, \\ 0, & \text{otherwise} \end{cases} \]

**Definition 1.2.** The matriceal Bloch space \( B(D, \ell^2) \) is the space of all analytic matrices \( A \) with \( A(r) \in B(\ell^2), 0 \leq r < 1 \), such that
\[
\|A\|_{B(D, \ell^2)} = \sup_{0 \leq r < 1} (1 - r^2) \|A'(r)\|_{B(\ell^2)} + \|A_0\|_{B(\ell^2)} < \infty
\]
where \( B(\ell^2) \) is the usual operator norm of the matrix \( A \) on the sequence space \( \ell^2 \) and \( A'(r) = \sum_{k=0}^{\infty} A_k kr^{k-1} \).

A matrix \( A \in B(D, \ell^2) \) is called a Bloch matrix. It is clear that the Toeplitz matrices which belong to the set of analytic matrices \( B(D, \ell^2) \) appears as an extension of the classical Bloch space of functions.

A very useful theorem is the following (see e.g. [13]):

**Theorem P** Both \( P : L^\infty(D, \ell^2) \to B(D, \ell^2) \) and \( P : \widetilde{L}^\infty(D, \ell^2) \to B(D, \ell^2) \) are bounded surjective operators.

The main results in this paper are presented, proved and discussed in Sections 2 and 3 below.

In Section 2 we give a characterization of matrices in the little Bloch space \( B_0(D, \ell^2) \) using the Bergman projection. One of the main results in this section is a new duality result (see Theorem 2.9). Also some related results are formulated and proved. In Section 3, we begin to state and prove with three technical lemmas, which are necessary to prove the main result of this section namely a characterization of the Bergman-Schatten spaces \( L_p^a(D, \ell^2) \), \( 1 \leq p \leq \infty \), in terms of Taylor coefficients (see Theorem 3.4).

2. A matrix version of the little Bloch space

Now we introduce another space of matrices, the so-called little Bloch space of matrices.

**Definition 2.1.** The space \( B_0(D, \ell^2) \) is the space of all upper triangular infinite matrices \( A \) such that \( \lim_{r \to 1-} (1 - r^2)\|(A \ast C(r))'\|_{B(\ell^2)} = 0 \), where \( C(r) \) is the Toeplitz matrix associated with the Cauchy kernel.

Clearly \( B_0(D, \ell^2) \) is a closed subspace of \( B(D, \ell^2) \) if the former is endowed with the norm of \( B(D, \ell^2) \).

We denote by \( E \) the Toeplitz matrix having all its entries equal to 1. First we state the following Lemma of independent interest:
Lemma 2.2. Let \( A \in B(D, \ell^2) \) and \( A_r(s) = A(rs) = A(r) * P(s) \) for all \( 0 \leq r < 1 \) and \( 0 \leq s < 1 \), where \( P(s) \) is the Toeplitz matrix associated to the Poisson kernel, that is
\[
P(s) = \begin{pmatrix}
1 & s_0 & s^2 & s^3 & \cdots \\
s & 1 & s & r^2_0 & \cdots \\
s^2 & s & 1 & s & \cdots \\
s^3 & s^2 & s & 1 & \cdots \\
\vdots & \vdots & \vdots & \ddots & \ddots \\
\end{pmatrix}.
\]

Then it follows that \( A_r \) is a matrix belonging to \( B_0(D, \ell^2) \) for all \( 0 \leq r < 1 \).

Proof. First we note that
\[
\lim_{s \to 1} (1 - s^2) \| A_r'(s) \|_{B(\ell^2)} = \lim_{s \to 1} (1 - s^2) r \| A'(rs) \|_{B(\ell^2)}
\]
and, by using well known facts about multipliers (see e.g. [5]) and elementary calculations, we find that
\[
\| A'(rs) \|_{B(\ell^2)} = \left\| \sum_{k=0}^{\infty} k A_k(r^k s^{k-1}) \right\|_{B(\ell^2)} = \left\| \sum_{k=0}^{\infty} k A_k r^{k-1} s^{k-1} \right\|_{B(\ell^2)}
\]

\[
= \| A'(r) \|_{B(\ell^2)} \cdot \left\| \sum_{k=0}^{\infty} s^{k-1} E_k \right\|_{M(\ell^2)} = \| A'(r) \|_{B(\ell^2)} \cdot \left\| \sum_{k=0}^{\infty} s^{k-1} e^{ik\theta} \right\|_{M(\ell^2)}
\]

\[
= \| A'(r) \|_{B(\ell^2)} \cdot \left\| \frac{1}{s} \int_{-\pi}^{\pi} \frac{1}{1 - s e^{i\theta}} d\theta \right\|_{L^1(\ell^2)} \leq \| A \|_{B(D, \ell^2)} \cdot \left\| \frac{1}{s(1 - r^2)} \int_{-\pi}^{\pi} \frac{1}{1 - s e^{i\theta}} d\theta \right\|_{L^1(\ell^2)}
\]

Thus, by making some straightforward calculations, we find that
\[
\lim_{s \to 1} (1 - s^2) \| A_r'(s) \|_{B(\ell^2)} \leq \| A \|_{B(D, \ell^2)} \cdot \frac{r}{1 - r^2} \lim_{s \to 1} (1 - s^2) \frac{1}{s}.
\]

\[
\int_{-\pi}^{\pi} \frac{1}{|1 - s e^{i\theta}|} d\theta = \| A \|_{B(D, \ell^2)} \cdot \frac{r}{1 - r^2} \lim_{s \to 1} (1 - s^2) \ln \frac{1}{1 - s} = 0.
\]

The proof is complete. \( \square \)

Our first result in this Section reads:
Theorem 2.3. Let $A \in B(D, \ell^2)$. Then $A \in B_0(D, \ell^2)$ if and only if $\lim_{r \to 1^-} \|A_r - A\|_{B(D, \ell^2)} = 0$.

Proof. By Lemma 2.2 it follows that $A_r \in B_0(D, \ell^2)$ and we use the fact that $B_0(D, \ell^2)$ is a closed subspace of $B(D, \ell^2)$ in order to conclude that the condition is sufficient.

Conversely, let $A \in B_0(D, \ell^2)$. Then, for every $\epsilon > 0$ there exists $0 < \delta < 1$ such that $(1 - s^2)\|A'(s)\|_{B(\ell^2)} < \epsilon$, for every $\delta^2 < s < 1$. We remark that

$$\|A_r - A\|_{B(D, \ell^2)} = \sup_{0 \leq s < 1} (1 - s^2)\|A'_r(s) - A'(s)\|_{B(\ell^2)} \leq$$

$$\leq \sup_{\delta < s < 1} (1 - s^2)\|A'_r(s) - A'(s)\|_{B(\ell^2)} + \sup_{0 \leq s \leq \delta} (1 - s^2)\|A'_r(s) - A'(s)\|_{B(\ell^2)}.$$

For $\delta < r < 1$ the first term is smaller than

$$(1 - r^2 s^2)\|A'(rs)\|_{B(\ell^2)} + (1 - s^2)\|A'(s)\|_{B(\ell^2)} < 2\epsilon.$$

The second term converges to 0 whenever $r \to 1^-$. Indeed, for $0 \leq s \leq \delta < \delta' < 1$, letting $u = \frac{s}{\delta'}$ and making some straightforward calculations, we get that

$$\|A'_r(s) - A'(s)\|_{B(\ell^2)} = \|rA'(rs) - A'(s)\|_{B(\ell^2)}$$

$$= \|rA'(s) \sum_{k=0}^{\infty} r^{k-1} E_k - A'(s)\|_{B(\ell^2)} = \|A'(s) \sum_{k=0}^{\infty} (r^{k-1}) E_k\|_{B(\ell^2)}$$

$$\leq \|A'(s)\|_{B(\ell^2)} \sum_{k=0}^{\infty} (r^{k-1}) E_k\|_{B(\ell^2)} \leq$$

$$\leq \|A'(s)\|_{B(\ell^2)} \sum_{k=0}^{\infty} (r^{k-1}) (\delta')^{k-1} E_k\|_{B(\ell^2)} \leq$$

$$\leq \|A'(s)\|_{B(\ell^2)} \int_{-1}^{1} \frac{1}{\pi |1 - r \delta' e^{i\theta}|} \frac{1}{|1 - \delta' e^{i\theta}|} \frac{d\theta}{2\pi} \leq$$

Then

$$\sup_{s \leq s} (1 - s^2)\|A'_r(s) - A'(s)\|_{B(\ell^2)} \leq$$

$$\leq \sup_{s \leq \delta, \delta' < 1} \left[ (1 - \frac{s}{\delta'})^2 \|A' \frac{s}{\delta'}\|_{B(\ell^2)} \cdot \frac{1 - s^2}{1 - \frac{s^2}{\delta'^2}} \right] \cdot \frac{(1 - r)}{(1 - \delta')(1 - \delta')} \leq$$

$$\leq \|A\|_{B(D, \ell^2)} \cdot \frac{1 - \delta^2}{1 - \frac{\delta^2}{\delta'^2}} \cdot \frac{(1 - r)}{(1 - \delta')(1 - \delta')}.$$
Consequently
\[ \lim_{r \to 1^{-}} \sup_{s \leq \delta} (1 - s^2) \| A'_r(s) - A'(s) \|_{B(\ell^2)} = 0, \]
i.e.
\[ \lim_{r \to 1^{-}} \| A_r - A \|_{B(D, \ell^2)} = 0. \]
The proof is complete. \(\square\)

**Corollary 2.4.** \( B_0(D, \ell^2) \) is the closure of all matrices of finite band type in the Bloch norm. In particular, this implies that \( B_0(D, \ell^2) \) is a separable space.

**Proof.** Let \( A \in B_0(D, \ell^2) \) and \( A^n = \sum_{k=0}^{n} A_k \). Then, by Theorem 2.3, it yields that for every \( \epsilon > 0 \) there is \( r_0 < 1 \) such that \( \| A r_0 - A \|_{B(D, \ell^2)} < \epsilon / 2. \)

We note that \( r \to A(r) \) for \( r \in [0, 1) \) is a continuous \( B(\ell^2) \)-valued function on \( [0, s] \) for \( s < 1. \)

Indeed, let \( 0 < s_n \leq s_0 < 1 \) and \( s_n \to s_0 \). Then,
\[ \| A(s_n) - A(s_0) \|_{B(\ell^2)} = \left\| \left[ C \left( \frac{s_n}{s} \right) - C \left( \frac{s_0}{s} \right) \right] * A(s') \right\|_{B(\ell^2)} \leq \]
\[ \leq \| A(s') \|_{B(\ell^2)} \left\| C \left( \frac{s_n}{s'} \right) - C \left( \frac{s_0}{s'} \right) \right\|_{M(\ell^2)}, \]
where \( s_n \to s_0 < s' < 1 \). Hence, by putting \( \delta = \frac{s_0}{s} < 1 \) and reasoning as in the proof of the previous theorem, we get that
\[ \| A(s_n) - A(s_0) \|_{B(\ell^2)} \leq \]
\[ \leq \| A(s') \|_{B(\ell^2)} \left\| \sum_{k=0}^{\infty} \delta^k \left( \left( \frac{s_n}{s_0} \right)^k - 1 \right) e^{ik\theta} \right\|_{M(\ell^2)} \to 0. \]
Now, for a fixed \( r_0 < 1 \) we have that
\[ \sup_{0 \leq s \leq 1} \| A r_0(s) \|_{B(\ell^2)} = M(r_0) = \| A(r_0) \|_{B(\ell^2)} < \infty \]
for all analytic matrices.

Thus, for \( r_0 < r' < 1 \) and by using the notation
\[ C^n \left( \frac{r_0}{r'} \right) = \sum_{k=0}^{n} C_k \left( \frac{r_0}{r'} \right), \]
we find that
\[ \| A r_0(\cdot) - (A r_0)^n(\cdot) \|_{L^\infty(D, \ell^2)} = \text{ess sup}_{s < 1} \| (A - A^n)(r_0 s) \|_{B(\ell^2)} = \]
\[ = \| (A - A^n)(r_0) \|_{B(\ell^2)} = \left\| C \left( \frac{r_0}{r'} \right) - C^n \left( \frac{r_0}{r'} \right) \right\| * A(r') \|_{B(\ell^2)} \leq \]
\[
\sum_{k=n+1}^{\infty} \left( \frac{r_0}{r^2} \right)^k e^{i \theta k} \cdot \|A(r')\|_{B(\ell^2)} = 0.
\]
Consequently,
\[
\|A_{r_0}(\cdot) - (A_{r_0})^n(\cdot)\|_{L^\infty(D, \ell^2)} \to 0
\]
and
\[
\|A(\cdot) - (A_{r_0})^n(\cdot)\|_{B(D, \ell^2)} \leq \epsilon,
\]
whenever \( r_0 < 1 \) is fixed as before and \( n \) is sufficiently large. Since \( \epsilon > 0 \) is arbitrary and \((A_{r_0})^n = \sum_{k=0}^{n} (A_{r_0})_k\) is a matrix of finite band type it follows that \( B_0(D, \ell^2) \) is the closure of all matrices of finite band type in the Bloch norm. The proof is complete.

The next theorem expresses a natural relation between the Bergman projection and the Bloch spaces. More exactly, our first main result in this Section is the following equivalence theorem:

**Theorem 2.5.** Let \( A \in B(D, \ell^2) \). Then the following assertions are equivalent:

1) \( A \in B_0(D, \ell^2) \).
2) There is a continuous \( B(\ell^2) \)-valued function \( r \to B(r) \) defined on \([0, 1]\) such that \( P(B(\cdot))(r) = A(r) \).
3) There is \( r \to B(r) \) which is a continuous \( B(\ell^2) \)-valued function such that \( \lim_{r \to 1} B(r) = 0 \) and satisfying \( P(B(\cdot))(r) = A(r) \).

**Proof.** To prove that 1) implies 3), let us take \( A \in B_0(D, \ell^2) \). We define \( A_1(r) := \sum_{k=2}^{\infty} A_k r^k \), with \( r < 1 \). Thus \( A_1'(r) = \sum_{k=2}^{\infty} k A_k r^{k-1} \) and \( A(r) = A_0 + A_1 r + A_1'(r) \). We take now
\[
B_2(r) = (1 - r^2)T * P(r) * A_1'(r),
\]
where \( T = (t_{ij})_{i,j} \) is a Schur multiplier which will be defined later on.

Thus, by the definition of the Bergman projection \( P \), we get that
\[
[P B_2(\cdot)](r)(i, j) = \begin{cases} t_{ij} a_{ij} (j-i+1)(j-i)r^{j-i} & \text{for } j-i \geq 2 \\ 0 & \text{otherwise} \end{cases}
\]
\[
= \begin{cases} 9 (j-i+1)(j-i) & \text{for all } j \neq i, i, j \geq 1 \\ 4(j-i) & \text{if } j = i, i \geq 1 \end{cases}
\]
Consequently, by taking \( t_{ij} = \frac{3(j-i+1)}{4(j-i)} \) for all \( j \neq i, i, j \geq 1 \), it follows that \( T \) is a Schur multiplier and \( P[B_2(\cdot)] = A_1(\cdot) \). Let now \( B(r) = 2(1 - r^2)A_0 + 3(1 - r^2)r A_1 + B_2(r) \). It is clear that \( [P B(\cdot)](r) = A(r) \). But, since \( A \in B_0(D, \ell^2) \),
it follows that $B_2(r)$ and, consequently, $B(r)$ is a continuous $B(\ell^2)$-valued function such that $\lim_{r \to 1} B(r) = 0$. Thus 3) holds.

It is obvious that 3) implies 2).

It remains to prove that 2) implies 1). Let 2) hold and choose $B(r) \in B(\ell^2)$ be such that

$$[PB(\cdot)](r) = A(r) \text{ for } r \in [0, 1].$$

Assume that $r \to B(r)$ is a continuous $B(\ell^2)$-valued function on $[0, 1]$ and let $M = \sup_{0 < r < 1} \|B(r)\|_{B(\ell^2)} < \infty$.

Let $0 < r_0 < 1$ be fixed and let us consider $A_{r_0}(r)$ given by the formula

$$A_{r_0}(r)(i, j) = \begin{cases} (j - i + 1)(rr_0)^{j-i}(2 \int_0^1 b_{ij}(s)s^{j-i+1}ds) & \text{if } j - i \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Consequently, according to Theorem P, we find that

$$A_{r_0}(\cdot) = P[P(r_0) \ast B(\cdot)] = P[B_{r_0}(\cdot)] \in B(D, \ell^2),$$

where $P(r_0)$ is the Toeplitz matrix associated with the Poisson kernel.

Let $C(\overline{D}, \ell^2)$ denote the space of all continuous $B(D, \ell^2)$-valued functions defined on $[0, 1]$. We will prove that the function $s \to P(P(r_0) \ast B(s))$ belongs to $C(\overline{D}, \ell^2)$ if $B$ is a continuous $B(\ell^2)$-valued function. Moreover, we will show that

$$\lim_{r \to 1} \sup_{s \in [0, 1]} \|A_r(s) - A(s)\|_{B(D, \ell^2)} = 0.$$ (1)

This, in its turn, implies that $\lim_{r \to 1} A_r = A$ in $B(D, \ell^2)$. Thus, by Theorem 2.3, it follows that $A \in B_0(D, \ell^2)$.

Let $s, s_0 \in [0, 1]$. Then

$$\|P(r_0) \ast [B(s) - B(s_0)]\|_{B(\ell^2)} \leq \left\| \sum_{k \in \mathbb{Z}} r_0^{|k|} e^{ik\theta} \|_{\mathcal{M}(\mathbb{T})} \cdot \|B(s) - B(s_0)\|_{B(\ell^2)} \to 0$$

for $s \to s_0$ and $B(s)$ is a continuous function on $[0, 1]$. Here we have used G. Bennett’s Theorem 1.1 and the fact that

$$\left\| \sum_{k \in \mathbb{Z}} r_0^{|k|} e^{ik\theta} \|_{\mathcal{M}(\mathbb{T})} = \left\| \frac{1 - r_0^2}{1 - r_0 e^{i\theta}} \right\|_{\mathcal{M}(\mathbb{T})} \leq 1.$$

Thus, the function $s \to P[P(r_0) \ast B(s)]$ belongs to $C(\overline{D}, \ell^2)$.

Hence, it only remains to prove that (1) holds.

$$\|P(r) \ast B(s) - B(s)\|_{B(\ell^2)} \leq \left\| \sum_{k \in \mathbb{Z}} (r^{|k|} - 1) e^{ik\theta} \|_{\mathcal{M}(\mathbb{T})} \cdot \|B(s)\|_{B(\ell^2)}$$

$$\leq M \cdot \left\| \sum_{k \in \mathbb{Z}} (r^{|k|} - 1) e^{ik\theta} \|_{\mathcal{M}(\mathbb{T})} \text{ for all } s \in [0, 1].$$
Denoting by $\mu_r(\theta)$ the measure $\sum_{k \in \mathbb{Z}} (r^{|k|} - 1) e^{ik\theta}$, then, for a trigonometric polynomial $\phi(\theta) = \sum_{m=-m}^{m} a_m e^{im\theta}$, we have that

$$
\mu_r(\phi) = \sum_{n=-m}^{m} (r^{|n|} - 1) a_n$$

and $|\mu_r(\phi)| \leq |\phi(r) - \phi(1)| \leq 2 \|\phi\|$, where $\phi(r)$ is the value of the Poisson extension of $\phi$ in the point $r$.

Consequently $\mu_r$ is a measure with a norm smaller than 2. But $\lim_{r \to 1} \mu_r(\phi) = 0$ for all trigonometric polynomials $\phi$. Thus $w^* - \lim_{r \to 1} \mu_r = 0$ in $M(T)$ and then is is clear that $\lim_{r \to 1} \|\mu_r\| = 0$ and by Theorem P the relation (1) is proved. Thus also the implication $2) \Rightarrow 1$ is proved and the proof is complete. \qed

By using similar ideas as in [13] we can obtain the following result:

**Theorem 2.6.** $P_2$ is a continuous operator (precisely a continuous projection) from $L^1(D, \ell^2)$ onto $L^1_a(D, \ell^2)$, where

$$
|P_2A(\cdot)(r)(i, j)| = \begin{cases} 
\frac{2^4 (j-i+4)!}{(j-i)!4!} r^{j-i} \int_0^1 (1 - s^2)^2 a_{ij}(s) s^{j-i} (2sds) & \text{if } j \geq i, \\
0 & \text{otherwise.}
\end{cases}
$$

**Proof.** The topological dual of $L^1(D, \ell^2)$ is $L^\infty(D, \ell^2)$ with respect to the duality pair:

$$
< A(\cdot), B(\cdot) > = \int_0^1 tr(A(s)|B(s)|^*) 2sds,
$$

where $A(\cdot) \in L^\infty(D, \ell^2)$, $B(\cdot) \in L^1(D, \ell^2)$ (see e.g. Theorem 8.18.2 in [10]). Using a duality argument it is sufficient to prove that $P_2^* : L^\infty(D, \ell^2) \to L^\infty(D, \ell^2)$ is bounded.

Now we are looking for the adjoint $P_2^*$ of $P_2$. We note that

$$
<P_2^* A(\cdot), B(\cdot) > = \int_0^1 \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} (P_2^* A(\cdot))(r)(i, j) \overline{b_{ij}(r)} (2rdr)
$$

and

$$
= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \int_0^1 (P_2^* A(\cdot))(r)(i, j) \overline{b_{ij}(r)} (2rdr).
$$

On the other hand it yields that

$$
<P_2^* A(\cdot), B(\cdot) > = < A(\cdot), P_2 B(\cdot) > = \int_0^1 trA(r) (P_2 B)^*(r)(2rdr)
$$

and

$$
= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \int_0^1 A(r)(i, j) (P_2 B)(r)(i, j) (2rdr)
$$
\[
\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{\Gamma(j-i+4)}{(j-i)!\Gamma(3)} \left( \int_0^1 [A(s)](i,j)s^{j-i}(2sds) \right) \times \left( \int_0^1 \frac{b_{ij}(s)s^{j-i}(1-s^2)^2 (2sds)}{\Gamma(j-i+4)(j-i)!\Gamma(3)} \right).
\]

Now let us consider \( \{I_k\} \), a sequence of intervals such that
\[
\lim_{k \to \infty} \mu(I_k) = 0, \quad d\mu = 2sds \text{ and } r \in I_k.
\]

For every \( k \) we take \( B(s)(i,j) = \chi_{I_k}(s)/\mu(I_k) \) and \( B(s)(l,k) = 0, (l,k) \neq (i,j) \) for every \((i,j) \in \mathbb{N} \times \mathbb{N} \).

By Lebesgue’s differentiation theorem (see e.g. [15]) we have that
\[
(P_2^* A(\cdot))(r)(i,j) =
\begin{cases}
\frac{\Gamma(j-i+4)}{(j-i)!\Gamma(3)} r^{j-i}(1-r^2)^2 \int_0^1 A(s)(i,j)s^{j-i}(2sds) & \text{if } j \geq i, \\
0 & \text{if } j < i,
\end{cases}
\]
a.e. for all \( r \in [0,1) \).

We will now prove that \( P_2^* : L^\infty(D,\ell^2) \to L^\infty(D,\ell^2) \) is a bounded operator. In order to prove that we first note that
\[
\|A(r)\|^2_{L^\infty(D,\ell^2)} = \text{ess sup}_{0 \leq r < 1} \|A(r)\|^2_{B(\ell^2)}
\]
\[
= \text{ess sup}_{0 \leq r < 1} \text{ sup } \sum_{j=1}^{\infty} |h_j|^2 \left( \sum_{i=1}^{\infty} a_{ij}(r)h_j \right)^2.
\]

Consequently,
\[
\|P_2^* A(\cdot)\|^2_{L^\infty(D,\ell^2)} =
\]
\[
= \text{ess sup}_{0 \leq r < 1} \text{ sup } \sum_{j=1}^{\infty} |h_j|^2 \left( \sum_{i=1}^{\infty} \frac{\Gamma(j-i+4)}{(j-i)!\Gamma(3)} r^{j-i}(1-r^2)^2 \right). \int_0^1 a_{ij}(s)s^{j-i}(2sds)^2
\]
\[
= \text{ess sup}_{0 \leq r < 1} \text{ sup } \sum_{j=1}^{\infty} |h_j|^2 \left( \sum_{i=1}^{\infty} \frac{\Gamma(j-i+4)}{(j-i)!\Gamma(3)} (2sds)^2 \right) \leq \text{ess sup}_{0 \leq r < 1} (1-r^2)^4. \int_0^1 \left( \sum_{j=1}^{\infty} a_{ij}(s) \right). \left( \frac{\Gamma(j-i+4)}{(j-i)!\Gamma(3)} h_j \right) (2sds)^2
\]
\[
\sup_{\|h\|_{\ell^2} \leq 1} \int_0^1 \left( \sum_{j=1}^{\infty} \left( \sum_{i=1}^{\infty} a_{ij}(s) \frac{\Gamma(j-i+4)}{(j-i)!\Gamma(3)} h_j \right)^2 (sds)^2 \right). \]
Since the Toeplitz matrix $C(rs) = ((c_{ij}(rs))_{i,j=1}^{\infty}$, where
\[ c_{ij}(rs) = c_{j-i}(rs) = \begin{cases} (rs)^{j-i}(j-i+3)(j-i+2)(j-i+1) & \text{if } j \geq i \\ 0 & \text{otherwise} \end{cases}, \]
is a Schur multiplier with

\[ \|C(rs)\|_{L^1(T)} = \|C(rs)\|_{L^1(T)} = 6 \sum_{n=0}^{\infty} (n+1)^2(rs)^{2n}, \]
we get that

\[ \sup_{\sum_{j=1}^{\infty} |h_j|^2 \leq 1} \left( \sum_{i=1}^{\infty} \sum_{j=i}^{\infty} a_{ij}(s)(rs)^{j-i}(j-i+3)(j-i+2)(j-i+1)h_j \right)^{1/2} = \|A(s) + C(rs)\|_{B(\ell^2)} \leq 6\|A(s)\|_{B(\ell^2)} \cdot \sum_{n=0}^{\infty} (n+1)^2(rs)^{2n}. \]

Consequently,

\[ \|P_2^* A(\cdot)\|^2_{L^\infty(D,\ell^2)} \leq \varepsilon \]
Proof. According to Theorem 2.5, for $B \in \mathcal{B}_0(D, \ell^2)$ we can find some $A(\cdot) \in \mathcal{C}_0(D, \ell^2)$ such that $[PA(\cdot)](r) = B(r)$. Clearly, we have that

$$P_2^n B = P_2^n PA = T^1 * (P_2^n A_1),$$

where $A_1(r) = T * A(r)$, for $T = (t_{j-i})_{i,j}$ with

$$t_{j-i} = \begin{cases} \frac{2(j-i+1)}{j-i+2} & \text{for } j - i \neq -2, \\ 0 & \text{otherwise} \end{cases}.$$  

$T$ is a Schur multiplier and the same is true for $T^1 = (t_{j-i}^1)_{i,j}$, where

$$t_{j-i}^1 = \begin{cases} \frac{j-i+2}{2(j-i+1)} & \text{for } j - i \neq 0, \\ 0 & \text{otherwise} \end{cases}.$$  

Thus, $\|A_1(r)\|_{B(\ell^2)} \sim \|A(r)\|_{B(\ell^2)}$ for all $r \in [0, 1]$. Hence, we obtain that

$$\|P_2^n A_1(r)\|_{B(\ell^2)}^2 =$$

$$= \sup_{\|h\|_{\ell^2} \leq 1} \sum_{i=1}^\infty \sum_{j=1}^\infty h_{ij} r^{j-i} \frac{\Gamma(j-i+4)}{(j-i)!} \frac{1}{2(j-i)!} \int_0^1 a_{ij}^1(s) s^{j-i}(2sds)^2 \leq$$

$$\leq \sup_{\|h\|_{\ell^2} \leq 1} (1-r^2)^4 \left[ \int_0^1 \left( \sum_{j=1}^\infty \sum_{i=1}^\infty a_{ij}^1(s) (rs)^{j-i} \frac{\Gamma(j-i+4)}{(j-i)!} h_{j}^2 \right)^2 (2sds)^2 \right] =$$

$$(1-r^2)^4 \sup_{\|h\|_{\ell^2} \leq 1} \left[ \int_0^1 \left( \sum_{i=1}^\infty \sum_{j=1}^\infty a_{ij}(r,s) (rs)^{j-i} (j-i+1)^2 (j-i+3) h_{j}^2 \right)^2 \right].$$

The Toeplitz matrix $C(r, s) = (c_{j-i}(r, s))_{i,j}$, where

$$c_{j-i}(r, s) = \begin{cases} (rs)^{j-i} (j-i+3) (j-i+1)^2 & \text{if } j \geq i, \\ 0 & \text{otherwise,} \end{cases}$$

is a Schur multiplier, since $\sum_{k=0}^\infty (rs)^k (k+3)(k+1)^2 e^{ik\theta} \sim \frac{1}{(1-r e^{i\theta})^2}$ and

$$\int_{-\pi}^{\pi} \frac{1}{(1-r e^{i\theta})^2} \frac{d\theta}{2\pi} \sim \sum_{n=0}^\infty n^2 (rs)^{2n}.$$  

Therefore, we have that

$$\|P_2^n A_1(r)\|_{B(\ell^2)} \leq C(1-r^2)^2 \int_0^1 \|A(s)\|_{B(\ell^2)} \cdot \left( \sum_{n=0}^\infty n^2 (rs)^{2n} \right) (2sds).$$

Since $\lim_{s \to 1} \|A(s)\|_{B(\ell^2)} = 0$, for all $\epsilon > 0$, there exists $\delta > 0$ such that $\|A(s)\| < \epsilon$ for all $s \geq \delta$ and, consequently,

$$\|P_2^n A_1(r)\|_{B(\ell^2)} \leq C(\epsilon + \|A(\cdot)\|_{C(D, \ell^2)}) \cdot \frac{(1-r^2)^2\delta^2}{(1-r^2\delta)^2}.$$
It follows that \( \lim_{r \to 1} \| P^*_2 A_1(r) \|_{B(\ell^2)} = 0 \) and since \( T^1 \) is a Schur multiplier it follows that \( P^*_2 B \in C_0(D, \ell^2) \) and \( \| P^*_2 B \|_{B(\ell^2)} \leq C \| A(\cdot) \|_{C(D, \ell^2)}. \)

Moreover, in view the proof of Theorem 2.5, we can find an \( A(\cdot) \in C_0(D, \ell^2) \) such that

\[
\| A(\cdot) \|_{C_0(D, \ell^2)} \leq C(\| B_0 \|_{B(\ell^2)} + \| B(\cdot) \|_{B(D, \ell^2)}),
\]

where \( C > 0 \) is an absolute constant. By now also using the arguments in the proof of Theorem (2.6) it follows that \( P^*_2 : B_0(D, \ell^2) \to C_0(D, \ell^2) \) is bounded.

On the other hand, if \( A \in B_0(D, \ell^2) \), then since \( A(r) \) is an analytic matrix it is obvious that \( A(r) = [PA(\cdot)](r) = (P[P^*_2 A(\cdot)])(r) \) for all \( r \in [0,1) \). Thus, by using Theorem P, we conclude that there exists a constant \( C > 0 \) such that \( \| A(\cdot) \|_{B(D, \ell^2)} \leq C \| P^*_2 A(\cdot) \|_{C(D, \ell^2)} \), which implies that \( P^*_2 : B_0(D, \ell^2) \to C_0(D, \ell^2) \) is an isomorphic embedding. The proof is complete. \( \square \)

**Remark 2.8.** From now on we identify \( B_0(D, \ell^2) \) with the space \( \tilde{B}_0(D, \ell^2) \) of all analytic matrices \( A * C(r) \) for \( A \in B_0(D, \ell^2) \).

We denote now by \( B_{0, c}(D, \ell^2) \) the closed Banach subspace of \( B_0(D, \ell^2) \) consisting of all upper triangular matrices whose diagonals are compact operators. We are now ready to prove that the little Bloch space \( B_{0, c}(D, \ell^2) \) in fact is the predual of the Bergman-Schatten space. More exactly, our last main result in this Section is the following duality result:

**Theorem 2.9.** It yields that \( B_{0, c}(D, \ell^2)^* = L_0^1(D, \ell^2) \) with respect to the usual duality, whenever \( B_0(D, \ell^2) \) is equipped with the norm induced by \( B(D, \ell^2) \).

**Proof.** Let \( A \in L_0^1(D, \ell^2) \). Then \( B \to \int_0^1 \text{tr} |B(s)A^*(s)|(2ds) \) defines a linear and bounded functional on \( B_{0, c}(D, \ell^2) \) (see e.g. Theorem 22 in [13]). Conversely, let us assume that \( F \) is a bounded linear functional on \( B_{0, c}(D, \ell^2) \). Then we shall prove that there is a matrix \( C \) from \( L_0^1(D, \ell^2) \) such that

\[
F(B) = \int_0^1 \text{tr} |B(r)C^*(r)|(2rdr),
\]

for \( B \) from a dense subset of \( B_0(D, \ell^2) \).

By Lemma 2.7 it follows that \( P^*_2 : B_0(D, \ell^2) \to C_0(D, \ell^2) \) is an isomorphic embedding. Thus \( X = P^*_2(B_{0, c}(D, \ell^2)) \) is a closed subspace in \( C_0(D, C_\infty) \) and \( F \circ (P^*_2)^{-1} : X \to \mathbb{C} \) is a bounded linear functional on \( X \), where \( C_0(D, C_\infty) \) is the subset in \( C_0(D, \ell^2) \) whose elements are \( C_\infty \)-valued functions. By the Hahn-Banach theorem \( F \circ (P^*_2)^{-1} \) can be extended to a bounded linear functional on \( C_0(D, C_\infty) \).

Let \( \Phi : C_0(D, C_\infty) \to \mathbb{C} \) denote this functional. It follows that \( C_0(D, C_\infty) = C_0[0,1] \tilde{\otimes}_e C_\infty \) and, thus, \( \Phi \) is a bilinear integral map, that is there is a bounded
Borel measure $\mu$ on $[0,1] \times U_{C_1}$, where $U_{C_1}$ is the unit ball of the space $C_1$ with the topology $\sigma(C_1, C_\infty)$, such that

$$\Phi(f \otimes A) = \int_{[0,1] \times U_{C_1}} f(r) \text{tr}(AB^*) d\mu(r, B)$$

for every $f \in C_0[0,1]$ and $A \in C_\infty$.

Thus, for the matrix $\sum_{k=0}^{n} A_k \in B_{0,c}(D, \ell^2)$, identified with the analytic matrix $\sum_{k=0}^{n} A_k r^k$, we have that

$$F(\sum_{k=0}^{n} A_k) = \int_{[0,1] \times U_{C_1}} \text{tr}((\sum_{k=0}^{n} (k+1)(k+2)(k+3) \frac{r^k}{2} (1 - r^2)^2 A_k)B^*) d\mu(r, B)$$

$$= \left< \mu(r, B), \text{tr}(\sum_{k=0}^{n} (k+1)(k+2)(k+3) \frac{r^k}{2} A_k)B^*(1 - r^2)^2 \right>.$$

On the other hand, we wish to have that

$$F(\sum_{k=0}^{n} A_k) = \int_{0}^{1} \text{tr}(\sum_{k=0}^{n} s^k A_k)(C(s)^*)(2sd)$$

$$= \int_{0}^{1} \text{tr}(\sum_{k=0}^{n} s^k A_k C_k^*)(2sd) = \sum_{k=0}^{n} \text{tr}A_k(\frac{C_k^*}{k+1}).$$

Therefore, letting $A = e_{i,i+k}$, denote the matrix having 1 as the single nonzero entry on the $i$th-row and the $(i+k)$th-column, for $i \geq 1$ and $j \geq 0$, we have that

$$C_k = \left< \mu(r, B), \frac{(k+1)(k+2)(k+3)}{2} r^k(1 - r^2)^2 B_k \right>, \ k = 0, 1, 2, \ldots.$$ 

Then, it yields that

$$\int_{0}^{1} ||C(s)||_{C_1}(2sd) =$$

$$= \int_{0}^{1} \int_{[0,1] \times U_{C_1}} \sum_{k=0}^{n} \frac{(k+1)(k+2)(k+3)}{2} (sr)^k(1 - r^2)^2 B_k.$$ 

$$d\mu(r, B)||_{C_1}(2sd) \leq \int_{[0,1] \times U_{C_1}} \left[ \int_{0}^{1} \sum_{k=0}^{n} \frac{(k+1)(k+2)(k+3)}{2} (rs)^k \right].$$
\[(1 - r^2)^2 B_k \|C_1(2sds)\|d\mu(r, B) \leq \int_{[0,1] \times U} \left[ \int_0^1 \| \sum_{k=0}^n \frac{(k+1)(k+2)}{2} \right] \]

\[(k + 3) (rs)^{k} (1 - r^2)^2 e^{ik(t)} \|B\|_L_1(\Sigma) \|B\|_C_1(2sds) \|d\mu(r, B) \leq \int_{[0,1] \times U} \left[ \int_0^1 \int_0^{2\pi} \frac{(1 - r^2)^2}{\left|1 - rs e^{it}\right|^4} (2sds) \|d\mu(r, B) \right] \]

\[\sim \int_{[0,1] \times U} \int_0^1 (1 - r^2)^2 \sum_{k=0}^\infty (n+1) (sr)^{2n} (2sds) \|d\mu(r, B) \]

\[= \int_{[0,1] \times U} (1 - r^2)^2 \sum_{n=0}^\infty (n+1) r^{2n} \|d\mu(r, B) = \|\mu\| < \infty.\]

Consequently, \(C \in L^p_{A}(D, \ell^2)\) and we get the relation (3) by using the fact that the set of all matrices \(\sum_{k=0}^n A_k\) is dense in \(B_{0,c}(D, \ell^2)\). The proof is complete. \(\square\)

3. A new characterization of Bergman-Schatten spaces

In this Section we give a characterization of the space \(L^p_{A}(D, \ell^2)\) in terms of Taylor coefficients which is similar to those obtained by M. Mateljevic and M. Pavlovic in [12]. For the proof of our main result (Theorem 3.4) we need the following three technical Lemmas.

**Lemma 3.1.** Let \(A = \sum_{k=0}^n A_k\), \(0 \leq m \leq n\). Then

\[\|A\|_{C_p} r^m \leq \|A(r)\|_{C_p} \leq \|A\|_{C_p} r^m, \ (0 < r < 1)\]

**Proof.** Let us take \(B[r] = \sum_{k=0}^n \mathcal{A}_r r^{n-k}\).

Then

\[\|A(r)\|_{C_p} = \|A(r)^*\|_{C_p} = \| \sum_{k=0}^n \mathcal{A}_r r^{k}\|_{C_p} = \|B[r]\|_{C_p} r^n.\]

Since \(\frac{1}{r} > 1\) and the function \(se^{it} \rightarrow \|B(se^{it})\|_{C_p}, \ 0 < s < 1, \ t \in [0,2\pi]\) is a subharmonic function, we have that

\[\|B[r]\|_{C_p} \left( \int_0^{2\pi} \left\|B[r] e^{it}\right\|_{C_p}^p \frac{dt}{2\pi} \right)^{\frac{1}{p}} \geq \left( \int_0^{2\pi} \|B(r)\|_{C_p}^p \frac{dt}{2\pi} \right)^{\frac{1}{p}} = \|B[1]\|_{C_p} = \|A^*\|_{C_p} = \|A\|_{C_p}.\]

This proves the left hand side inequality. The proof of the right hand side inequality is similar so we omit the details. \(\square\)
Lemma 3.2. Let $A$ be an upper triangular matrix,

$$
\sigma_k(A) = \sum_{k=0}^{n} \left( 1 - \frac{k}{n+1} \right) A_k,
$$

the Cesaro mean of the order $k$ and

$$
\|\sigma_n(A)\|_p = \sup_{0 < r < 1} \|\sigma_n(A)(r)\|_{C_p}, \ (n = 0, 1, 2, \ldots).
$$

Then

$$
\|\sigma_k(A)\|_p r^k \leq \|A(r)\|_{C_p} \leq (1 - r)^2 \sum_{n=0}^{\infty} \|\sigma_n(A)\|_p (n+1) r^n.
$$

Proof. First we observe that

$$
\|A(r)\|_{C_p} \geq \|\sigma_n(A)(r)\|_{C_p}, \text{ for every } r \in (0, 1), n \in \mathbb{N}.
$$

Since

$$
\left\| \sum_{k=-n}^{n} \left( 1 - \frac{|k|}{n+1} \right) e^{i k t} \right\|_{L^1(T)} \leq 1,
$$

$F_n \in M(\ell^2) = M(C_1) \subset M(C_p) \subset M(C_2)$ for $1 \leq p \leq 2$ and $M(C_p) = M(C_p')$, $\frac{1}{p} + \frac{1}{p'} = 1$, for $2 \leq p < \infty$ it follows that

$$
\|\sigma_n(A)(r)\|_{C_p} = \|A(r) * F_n\|_{C_p} \leq \|A(r)\|_{C_p}.
$$

By using this inequality and Lemma 3.1 we find that

$$
\|A(r)\|_{C_p} \geq \|\sigma_n(A)(r)\|_{C_p} \geq r^n \|\sigma_n(A)\|_{C_p} \geq r^n \|\sigma_n(A)\|_p,
$$

and the left hand side of the inequality is proved. The proof of the right hand of the inequality follows by using the formula

$$
A(r) = (1 - r)^2 \sum_{n=0}^{\infty} \sigma_n(A)(n+1) r^n
$$

and Minkovski’s inequality. The proof is complete. \qed

Lemma 3.3. Let $A$ be an upper triangular matrix, $0 \leq k < n$ and $p \geq 1$. Then we have that

$$
(n-k+1)\|\sigma_k(A)\|_p \leq (n+1)\|\sigma_n(A)\|_p.
$$

Proof. First we note that $\|\sigma_n(A)\|_p = \sup_{0 < r < 1} \|\sigma_n(A)(r)\|_{C_p}$. Since

$$
\|\sigma_k(B)\|_{C_p} \leq \|B\|_{C_p}
$$

it follows that

$$
\|\sigma_n(A)\|_p \geq \|\sigma_k \sigma_n(A)\|_p = \|\sigma_k(A) - \frac{r}{n+1} \sigma_k'(A)\|_p
$$
\[ \geq \|\sigma_k(A)\|_p - \frac{1}{n+1}\|\sigma_k'(A)\|_p \geq \]
[by Bernstein’s inequality]
\[ \geq \|\sigma_k(A)\|_p - \frac{k}{n+1}\|\sigma_k(A)\|_p, \]
and the inequality (4) is proved. \qed

Our main result in this Section reads:

**Theorem 3.4.** Let \( A \) be an analytic matrix. Then \( A \in L^p_0(D, \ell^2) \) if and only if
\[ \sum_{n=0}^{\infty} \frac{1}{(n+1)^2} \|\sigma_n(A)\|_p^p < \infty. \]

**Proof.** First we prove that \( A \in L^p_0(D, \ell^2) \) if and only if
\[ \left( \int_0^1 \|A(r)\|_{C_p}^p \, dr \right)^{\frac{1}{p}} < \infty. \]
If \( \left( \int_0^1 \|A(r)\|_{C_p}^p \, dr \right)^{\frac{1}{p}} < \infty \), then it is clear that \( A \in L^p_0(D, \ell^2) \). Conversely if \( A \in L^p_0(D, \ell^2) \), then \( \|A(r)\|_{L^p(D, \ell^2)} < \infty \).

Let \( E_\theta \) be the Toeplitz matrix corresponding to the Dirac measure \( \delta_\theta \), i.e.,
\[ E_\theta = (e^{ij})_{k,j \geq 1}, e_{kj} = e^{i(j-k)t}. \]
Then \( E_\theta \in M(\ell^2) = M(C_1) \subset M(C_p), 1 \leq p < \infty. \)
Since \( re^{i\theta} \to \|A(r) * E_\theta\|_{C_p} \) is subharmonic on \( D \) it follows that \( \int_0^{2\pi} \|A(r) * E_\theta\|_{C_p}^p \, d\theta \leq \int_0^{2\pi} \|A(\sqrt{r} * E_\theta)\|_{C_p}^p \, d\theta \). Therefore
\[ \int_0^1 \int_0^{2\pi} \|A(r) * E_\theta\|_{C_p}^p \, d\theta \, dr \leq \int_0^1 \int_0^{2\pi} \|A(\sqrt{r} * E_\theta)\|_{C_p}^p \, d\theta \, dr \]
and
\[ \int_0^1 \|A(r)\|_{C_p}^p \, dr \leq \int_0^1 \|A(\sqrt{r})\|_{C_p}^p \, dr = 2 \int_0^1 \|A(s)\|_{C_p}^p \, ds < \infty, \]
and (5) is proved.

Now let \( A \in L^p_0(D, \ell^2) \). Then, by the first inequality in Lemma 3.2, we have that
\[ \|A(r)\|_{C_p}^p = (1-r) \sum_{n=0}^{\infty} \|A(r)\|_{C_p}^p r^n \geq (1-r) \sum_{n=0}^{\infty} \|\sigma_n(A)\|_{p}^n (p+1). \]
Now integration yields that
\[ \infty > \int_0^1 \| A(r) \|_C^p \, dr \geq \int_0^1 (1-r) \sum_{n=0}^{\infty} \| \sigma_n(A) \|_p n^{(p+1)} \, dr \]
\[ \geq C^{-1} \sum_{n=0}^{\infty} \frac{1}{(n+1)^2} \| \sigma_n(A) \|_p. \]

Conversely, suppose that
\[ \sum_{n=0}^{\infty} \frac{\| \sigma_n(A) \|_p^p}{(n+1)^2} < \infty. \]

Let \( x_n = \sum_{k=0}^{\infty}(k+1)(n-k+1)\| \sigma_k(A) \|_p \). Then, by summing by parts as in (4.4) in [12], we find that
\[ \sum_{n=0}^{\infty} \| \sigma_n(A) \|_p (n+1) r^n = (1-r)^2 \sum_{n=0}^{\infty} x_n r^n. \]

On the other hand, by using Lemma 3.3, we see that
\[ x_n \leq C(n+1)^3 \| \sigma_n(A) \|_p \]
and, therefore,
\[ \sum_{n=0}^{\infty} \frac{1}{(n+1)^{3p+2}} x_n^p < \infty. \]

We use now Lemma 4.8 in [12] with \( q = p, \phi(r) = r^\frac{2}{p}, r \in (0,1] \) and obtain that
\[ \int_0^1 \left[ (1-r)^4 \sum_{n=0}^{\infty} x_n r^n \right]^p \, dr < \infty. \]

Moreover by using also (6) we arrive at
\[ \int_0^1 (1-r)^{2p} \left( \frac{1}{(1-r)^2} \sum_{n=0}^{\infty} \| \sigma_n(A) \|_p (n+1) r^n \right)^p \, dr < \infty. \]

It follows that
\[ \int_0^1 (1-r)^{2p} \left( \sum_{n=0}^{\infty} \| \sigma_n(A) \|_p (n+1) r^n \right)^p \, dr < \infty. \]

From the right hand side inequality in Lemma 3.2, we finally obtain that
\[ \int_0^1 \| A(r) \|_C^p \, dr \leq \int_0^1 (1-r)^{2p} \left( \sum_{n=0}^{\infty} \| \sigma_n(A) \|_p (n+1) r^n \right)^p \, dr < \infty. \]

The proof is complete. \( \square \)
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References


