Optimal Power Allocation and Ergodic Capacity in Cognitive Radio Network

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Abstract

Cognitive radio network allows the secondary users (SUs) in a secondary communication network (SCN) coexist with priority users in a priority communication network (PCN). The interference caused by SCN to PCN is required to be rational and not affected the quality of server within the PCN to each priority users (PUs). In this paper, we study the optimal power allocation method to achieve the maximum ergodic capacity of both PU fading channel and SU fading channel under different power constraints. The interference power constraints at PU and the transmit power constraints at SU are considered, and the power constraints is distinguished as the peak power constraint and average power constraint. Different combinations of power constraints are analyzed under the particular system and channel fading model. This paper shows that in different scenarios the optimal power of SCN do exist to reach the maximum ergodic capacity of the entire system including SCN and PCN.
Sammanfattning

Kognitiv radio nätverk tillåter sekundära användare (SUs) i ett sekundärt kommunikationsnät (SCN) att samexistera med prioriterade användare i ett prioriterat kommunikationsnätverk (PCN). Den störning som orsakas av SCN till PCN krävs för att vara rationell och inte påverka kvaliteten på servern inom PCN till varje prioriterad användare (PUs). I denna uppsats studerar vi optimal effektfördelningsmetod för att uppnå maximal ergodisk kapacitet av både PU fading kanal och SU fading kanal under olik kraftbegränsningar. De störningskraftbegränsningar i PU och sändningskraft begränsningar vid SU beaktas, och kraftbegränsningar är identifierat som toppoeffektbegränsning och medel-effektbegränsning. Olik kombinationer av kraftbegränsningar analyseras i det särskilda systemet och kanal fading modellen. Detta dokument visar att olik scenarion existerar där den optima kraften i SCN når maximal ergodisk kapacitet av hela systemet inklusive SCN och PCN.
# Contents

1 Introduction ........................................... 1
   1.1 Cognitive radio networks .......................... 1
   1.2 Literature review of spectrum sharing .................. 2
   1.3 Thesis project work .................................. 3

2 Cognitive Radio Network Model ......................... 4
   2.1 System model ........................................ 4
      2.1.1 Parameters definition ........................... 4
      2.1.2 Description of power constraint .................. 6
      2.1.3 Summary .......................................... 7

3 Ergodic Capacity ...................................... 8
   3.1 Peak transmit power constraint and peak interference power constraint .... 8
   3.2 Peak transmit power constraint and average interference power constraint ... 11
   3.3 Average transmit power constraint and peak interference power constraint ... 14
   3.4 Average transmit power constraint and average interference power constraint ... 15

4 Numerical Results .................................... 16
   4.1 Simulation results ................................... 16
   4.2 Summary ............................................ 20

5 Conclusion ........................................... 22
Chapter 1

Introduction

1.1 Cognitive radio networks

The spectrum of radio is important for wireless communication networks. As for the radio spectrum is increasing crowded these years, it was considered as a precious resource to wireless communication services. Many promising techniques were developed to solve the challenge problem between increasing communication process and the limitation of radio spectrum recourse [1], [2], and a more efficient spectrum usage method was required [3]. Although the limitation of physical spectrum resource was the biggest restricted condition to limit the amount of communication processes, research indicate that the inefficient and inflexible of allocating spectrum policy is the main reason that caused the problem of the spectrum efficiency.

Cognitive radio (CR) [4] is a promising technology to deal with the spectrum under-utilization problem caused by the current inflexible spectrum allocation policy. There are two popular ideas to realize a cognitive radio network (CRN), the first technique is called as opportunistic spectrum access [5] and the second technique is the spectrum sharing [6]. For opportunistic spectrum access, in a CRN, a secondary user (SU) in the secondary communication network (SCN) is allowed to access the spectrum that is originally allocated to the primary users (PUs) when the spectrum is not used by any PU. In this way, the spectrum utilization efficiency can be greatly improved. However, to precisely detect a vacant spectrum is not an easy task [7]. To achieve these functions, intelligent detection techniques have been adopted in CR so that the SUs can sense, learn, and adapt to the dynamic network conditions.

As for the spectrum sharing, we noticed that one of the important requirements of CRNs is that SUs must minimize harmful interference to PUs. CRN can also be designed to allow simultaneous transmission of PUs and SUs. From PUs perspective, SU is allowed to transmit as long as the interference from SU does not degrade the quality of service (QoS) of PU to an unacceptable level. From SUs perspective, SU should control its transmit power properly in order to achieve a reasonably high transmission rate without causing too much interference to PU. Traditionally, the capacity of fading channels is studied under various transmit power constraints, and the corresponding optimal and sub-optimal power allocation policies are given in, e.g., [8], [9], [10].
1.2 Literature review of spectrum sharing

Recently, study on the channel capacity of SU link under spectrum sharing has attracted a lot of attention. Specifically, SU channel capacity under spectrum sharing was addressed by Gastpar in [11], where the capacities of different additive white Gaussian noise (AWGN) channels are derived under a received power constraint. The capacities derived in [11] are shown to be quite similar to those under a transmit power constraint. This is non-surprising because the ratio of the received power to the transmit power is fixed in an AWGN channel; thus, considering a received power constraint is equivalent to considering a transmit power constraint. However, in the presence of fading, the situation becomes quite different. In [7], the authors derived the optimal power allocation strategy for a SU coexisting with a PU subject to an interference power constraint at PU receiver, and evaluated the ergodic capacity (the maximum on the long-term average rate that can be achieved by fading channel, a capacity metric that is suitable for delay-insensitive applications) for SU channel for different fading channel models. In [12], the authors considered the outage capacity under both the peak and the average interference power constraints. It is noted that optimal design of SU transmission strategy under interference power constraints at PU receivers has also been studied in [13] for multi-antenna CR transmitters, and in [14] for multiple CR transmitters in a multiple-access channel (MAC). For a block fading (BF) channel [15], [16], the channel remains constant during each transmission block, but possibly changes from one block to another. Ergodic capacity is a good performance limit indicator for delay-insensitive services, when the codeword length can be sufficiently long to span over all the fading blocks. However, for real-time applications, it is more appropriate to consider the delay-limited capacity introduced in [17], which is defined as the maximum constant transmission rate achievable over each of the fading blocks.

Also, there has been an increasing interest in defining the fundamental limits on the rates that spectrum-sharing channels can achieve. In this context, [18] proposed a protocol in which the primary and secondary users are allowed to simultaneously transmit over the same time or frequency and obtained the capacity of an additive white Gaussian noise (AWGN) channel in this configuration. In addition, [19] characterizes the maximum achievable rate of the secondary user provided with knowledge of the primary users codeword under mutual interference from the primary transmission. A Comprehensive overview of the fundamental limits, policy issues and challenges of spectrum sharing channels can also be found in [20]. In a spectrum-sharing environment, the secondary user may opportunistically transmit with high input powers as long as the interference caused by its transmission onto the primary receiver remains below a predefined threshold. Hence, imposing a constraint on the received-power at a third user partys receiver, e.g., primary receiver, may be a more relevant constraint than a maximum on the transmit power. In this respect, [11] investigated the capacity of an AWGN channel under received-power constraint, and showed that in a point-to-point channel, the transmitted- and received-power constraints result in similar capacity formulae. Later, the ergodic capacity of a point-to-point system in fading channels considering either peak or average received-power constraints at a third party receiver was studied in [6]. This work showed that significant capacity gains may be achieved if the channels are time-varying due to fading. This is in contrast to systems with transmit power...
constraints, where capacity is known to degrade because of fading [8].

1.3 Thesis project work

In this thesis, we only consider the spectrum sharing. We focus on the performance of the secondary user when accessing the primary network under four different power constraints, and consider metrics of the ergodic capacity and delay limited capacity within three different channel models. In this thesis, we derive the optimal power allocation strategies for SU to achieve aforementioned capacities. Besides the interference power constraint to protect PU, we also consider the transmit power constraint of SU transmitter. Since the transmit power and the interference power can be limited either by a peak or an average constraint, different combinations of power constraints are considered. It is shown that there ergodic capacity gain for SU under the average over the peak transmit interference power constraint.

The remaining of this thesis report is organized as follows: in Chapter 2, the cognitive radio network was modeled; in Chapter 3, the ergodic capacity was calculated and analyzed within the considered model; in Chapter 4, the simulation results were presented and discussed; in Chapter 5, the conclusion and future work were given.
Chapter 2

Cognitive Radio Network Model

In this chapter we would introduce the system model which would be investigated. And the optimal function could be formulated based on the system model that we studied.

2.1 System model

As illustrated in Figure 2.1, we consider a spectrum sharing network with one PU and one SU. The link between SU transmitter (SU-Tx) and PU receiver (PU-Rx) is assumed to be a flat fading channel with instantaneous channel power gain $g_0$ and the AWGN $n_0$. SU channel between SU-Tx and SU receiver (SU-Rx) is also a flat fading channel characterized by instantaneous channel power gain $g_1$ and the AWGN $n_1$. The link between PU transmitter (PU-Tx) and PU-Rx is a flat fading channel characterized by instantaneous channel power gain $g_2$ and the AWGN $n_2$. The noises $n_0$ and $n_1$ are assumed to be independent random variables with the distribution $CN(0, N_0)$ (circularly symmetric complex Gaussian variable with mean zero and variance $N_0$). Perfect channel state information (CSI) on $g_0$, $g_1$, and $g_2$ is assumed to be available at SU-Tx. Furthermore, it is assumed that the interference from PU-Tx to SU-Rx can be ignored at SU-Rx, because the received signal power from PU-Tx is below the transmission power threshold of the SU-Rx. System model is shown in Figure 2.1.

2.1.1 Parameters definition

For we focus on the performance of SU, we defined the interference channel gain as $g_0$ which was the channel between the PU and the secondary base station (BS); the transmit channel gain as $g_1$ which was the channel between the SU and the secondary BS; the transmit channel gain as $g_2$ which was the channel between the PU and the primary channel, the power of AWGN is defined as $N_0$. The transmit power from secondary network was defined as $P(g_0, g_1, g_2)$, which was determined by the channel gains. And the transmit power from the primary network was defined as $P_B$. Since we considered four basic power constraints.
which constraints the behavior of SU’s. The four constraints were showed below: $P_{pk}$ was indicated as the peak transmit power constraint; $P_{av}$ was the indicated as the average power constraint; $Q_{pk}$ was indicated as the peak interference power constraint; $Q_{av}$ was indicated as the average interference power constraint. Within those four basic constraints, we could get four considerable combinations of four constraints. The first constraint was the peak transmit power combined with the peak interference power constraint, which defined as $F_1$; the second constraint was the peak transmit power combined with the average interference power constraint, which defined as $F_2$; the third constraint was the average transmit power constraint combined with the peak interference power, which defined as $F_3$; the fourth constraint was the average power constraint combined with the average power constraint, which defined as $F_4$. In this paper we would consider those four constraints during different communication channels and analyzed the ergodic capacity and the performance in order to manage the optimal power allocation for secondary user network. We assume that the power of overall network including the primary network and secondary network is the same, and $\alpha, 1 - \alpha$ were the constant weight value in order to indicate the priority level between primary network and secondary network. Here we describe the optimize function to solve the ergodic capacity problem by maximize the achievable rate of both primary network and secondary network.
The optimization function was showed below:

\[
\max_{P(g_0,g_1,g_2) \in f} E[\alpha \log_2(1 + \frac{g_1 P(g_0,g_1,g_2)}{N_0}) + (1 - \alpha) \log_2(1 + \frac{g_2 P_B}{N_0 + g_0 P(g_0,g_1,g_2)})]
\]

(2.1)

Where \( f \in (F_1,F_2,F_3,F_4) \), which were the four constraints that mentioned in previous section. For we had to optimize the function under those four constraints respectively to find out the result of the optimal \( P(g_0,g_1,g_2) \) and analyses the ergodic capacity and outage capacity performance of the optimal \( P(g_0,g_1,g_2) \) under different channel model.

2.1.2 Description of power constraint

As we defined the combination of four basic power constraints form the secondary network aspect which were the peak transmit power constraint; the average transmit power constraint; the peak interference power constraint and the average interference power constraint. We defined \( F_1, F_2, F_3 \) and \( F_4 \) which were the combination of those basic constraint as well. In this section we would look into these constraints and discuss the concepts and meaning of them. In the previous section we have considered two types of power constraints: the peak power constraint and the average power constraint. The peak power limitation may be due to the nonlinearity of power amplifiers in practice, while the average power is restricted below a certain level to keep the long-term power budget. The \( P(g_0,g_1,g_2) \) was the transmit power in secondary network which related to the channel gain pair \( P(g_0,g_1,g_2) \). And the transmit power were shown as follows:

\[
P(g_0,g_1,g_2) \geq 0, \forall (g_0,g_1,g_2)
\]

(2.2)

For the limitation of the peak transmit power constraint and the average transmit power constraint can be represented as:

\[
P(g_0,g_1,g_2) \leq P_{pk}, \forall (g_0,g_1,g_2)
\]

(2.3)

\[
E[P(g_0,g_1,g_2)] \leq P_{av}, \forall (g_0,g_1,g_2)
\]

(2.4)

For the interference power in the secondary user network, which was the transmit power form the primary user network aspect, they affected the QoS of primary user network. According to the researches which investigate the channel capacities with received power constraints. If the primary user network need the instantaneous QoS requirement, the peak interference power should be considered. As for the average interference power, which can ensure the long-term QoS of primary user network. The limitation of the peak interference power constraint and the average interference power constraint which can be represented as follow:

\[
P(g_0,g_1,g_2) \leq Q_{pk}, \forall (g_0,g_1,g_2)
\]

(2.5)

\[
E[P(g_0,g_1,g_2)] \leq Q_{av}, \forall (g_0,g_1,g_2)
\]

(2.6)
As we had mentioned in the previous section, in order to analyses the performance of secondary user network under these power constraints, the transmit power constraints and the interference power constraints were combined into four sets of power constraints. The four sets of power constraints which we considered through all our study were:

\[
F_1 = P(g_0, g_1, g_2) : (1), (2), (3) \quad (2.7)
\]
\[
F_2 = P(g_0, g_1, g_2) : (1), (2), (5) \quad (2.8)
\]
\[
F_3 = P(g_0, g_1, g_2) : (1), (3), (4) \quad (2.9)
\]
\[
F_4 = P(g_0, g_1, g_2) : (1), (3), (5) \quad (2.10)
\]

Where \( F_1 \) was the combination of peak transmit power constraint and peak interference power constraint; \( F_2 \) was the combination of peak transmit power constraint and average interference power constraint; \( F_3 \) was the combination of average transmit power constraint and peak interference power constraint; \( F_4 \) was the combination of average transmit power and average interference power constraint.

2.1.3 Summary

In this chapter, we introduced the cognitive radio network model and formulated the optimization function which will be used to calculate the ergodic capacity. We also described and discussed the combination of four different power constraints. In the next chapter, we will derive the ergodic capacity based on the formulated optimization function and the four different constraints.
Chapter 3

Ergodic Capacity

In this chapter, we would introduce the methods and algorithm to solve our optimization question. The four constraints combination scenarios would be discussed respectively. And the optimal result of $P(g_0, g_1, g_2)$ would be calculated to achieve the maximum ergodic capacity of the entire system.

3.1 Peak transmit power constraint and peak interference power constraint

First of all we would discuss the basic scenario of our system model, which was calculated and analyzed under the peak transmit power and peak interference power constraints. In this scenario, quadratic function and the extreme values were calculated within a particular range. The existence of the maximum $P(g_0, g_1, g_2)$ would be discussed as well according to the attributes of the quadratic function. The optimal function was shown as follow:

$$\max_{P(g_0, g_1, g_2) \in F_1} E[\alpha \log_2(1 + \frac{g_1 P(g_0, g_1, g_2)}{N_0}) + (1 - \alpha) \log_2(1 + \frac{g_2 P_B}{N_0 + g_0 P(g_0, g_1, g_2)})] \tag{3.1}$$

Subject to:

$$\begin{cases} P(g_0, g_1, g_2) \geq 0 \\ P(g_0, g_1, g_2) \leq P_{pk} \\ P(g_0, g_1, g_2) \leq Q_{pk} \end{cases} \tag{3.2}$$

Where $f$ here was the constraints combination which showed as $F_1$. From the constraints we could obtain that:

$$0 \leq P(g_0, g_1, g_2) \leq \min \left\{ P_{pk}, \frac{Q_{pk}}{g_0} \right\} \tag{3.3}$$

Considering the the optimal function, the calculation of expectation would be simplified as:

$$\max_{P(g_0, g_1, g_2) \in F_1} \alpha \log_2(1 + \frac{g_1 P(g_0, g_1, g_2)}{N_0}) + (1 - \alpha) \log_2(1 + \frac{g_2 P_B}{N_0 + g_0 P(g_0, g_1, g_2)}) \tag{3.4}$$
Assume that:

\[
\begin{aligned}
P(g_0, g_1, g_2) &= 0 \\
\frac{1}{n^2} &= k
\end{aligned}
\quad (3.5)
\]

Then the function \(\text{(3.4)}\) would be expressed as:

\[
F(x) = \alpha \log_2 (1 + \frac{g_1 x}{N_0}) + (1 - \alpha) \log_2 (1 + \frac{g_2 P_B}{N_0 + g_0 x})
\quad (3.6)
\]

In order to find the maximum value of \(F(x)\), the extreme values were needed to calculate at first. Take the derivation of function \(F(x)\) \(3.6\) and let it equal to zero to calculate the extreme values. And we could assume that:

\[
f(x) = F(x)' = \alpha \log_2 \left(1 + \frac{\frac{g_1}{N_0}}{\frac{g_1}{N_0} + \ln 2}\right) + (1 - \alpha) \log_2 \left(1 + \frac{\frac{g_2 P_B}{N_0 + g_0 x}}{\frac{g_2 P_B}{N_0 + g_0 x}}\right)
\quad (3.7)
\]

\[
f(x) = \alpha \left(\frac{k g_1}{N_0 + g_1 x}\right) + (1 - \alpha) \left(\frac{-g_0 g_2 P_B k}{(N_0 + g_0 x)^2 + (N_0 + g_0 x) g_2 P_B}\right)
\quad (3.8)
\]

\[
\text{In order to find the maximum value of optimal function we have to find the extreme points of which were the zero point of the derivation function. Let the function equal to 0 and we could get following equation:}
\quad (3.9)
\]

\[
\alpha \left(\frac{k g_1}{N_0 + g_1 x}\right) + (1 - \alpha) \left(\frac{-g_0 g_2 P_B k}{(N_0 + g_0 x)^2 + (N_0 + g_0 x) g_2 P_B}\right) = 0
\quad (3.10)
\]

Then we simplified the equation \(3.10\), we could get:

\[
(\alpha g_0^2 g_1 k) x^2 + (2\alpha g_0 g_1 k N_0 - g_0 g_1 g_2 k P_B + 2\alpha g_0 g_1 g_2 k P_B) x
\quad (3.11)
\]

\[
+ (\alpha g_1 k N_0^2 - g_0 g_2 k N_0 P_B + \alpha g_0 g_2 k N_0 P_B + \alpha g_1 g_2 k N_0 P_B) = 0
\]

We noticed that \(x\) is non-negative. That means the range of the weight \(\alpha\) is constrained by this condition as well. Then we would know this equation has the same structure of basic quadratic equation, which was

\[
f(x) = ax^2 + bx + c, \quad a \neq 0
\quad (3.12)
\]

And we could get:

\[
\begin{cases}
\Delta = b^2 - 4ac \\
X_1 = \frac{-b + \sqrt{\Delta}}{2a} \\
X_2 = \frac{-b - \sqrt{\Delta}}{2a}
\end{cases}
\quad (3.13)
\]

Where

\[
\begin{aligned}
a &= \alpha g_0^2 g_1 k \\
b &= 2\alpha g_0 g_1 k N_0 - g_0 g_1 g_2 k P_B + 2\alpha g_0 g_1 g_2 k P_B \\
c &= \alpha g_1 k N_0^2 - g_0 g_2 k N_0 P_B + \alpha g_0 g_2 k N_0 P_B + \alpha g_1 g_2 k N_0 P_B
\end{aligned}
\quad (3.14)
\]
From (3.14) we could have that $a = 0$ if and only if $\alpha = 0$, and the optimal $x$ could be calculated as:

$$P(g_0, g_1, g_2) = -\frac{2\alpha g_0 g_1 kN_0 - g_0 g_1 g_2 kP_B + 2\alpha g_0 g_1 g_2 kP_B}{\alpha g_1 kN_0^2 - g_0 g_2 kN_0 P_B + \alpha g_0 g_2 kN_0 P_B + \alpha g_1 g_2 kN_0 P_B}$$ (3.15)

We would discuss the situations when $a$ not equal to zero. As for $a$ was positive value for $\alpha$ is the weight value ranged from zero to one, and $g_0$, $g_1$ were the random variable which obey the Rayleigh distribution. And the discussions were presented as follow:

If

$$\Delta \leq 0, \text{or } \begin{cases} \Delta > 0, \\
\max \{X_1, X_2\} \leq 0, \end{cases} \text{or } \begin{cases} \Delta > 0, \\
\min \left\{P_{pk}, \frac{Q_{pk}}{g_0}\right\} \leq \min \{X_1, X_2\} \end{cases}$$ (3.16)

$F(x)$ (3.6) was monotonically increasing with the range $[0, \min \left\{P_{pk}, \frac{Q_{pk}}{g_0}\right\}]$. In case (3.16) we could get

$$P(g_0, g_1, g_2) = \min \left\{P_{pk}, \frac{Q_{pk}}{g_0}\right\}$$ (3.17)

Since $F(x)$ would have the maximum value when $P(g_0, g_1, g_2)$ equals to the maximum value within the definitional domain.

If:

$$\begin{cases} \Delta > 0, \\
\min \{X_1, X_2\} \leq 0, \\
0 < \max \{X_1, X_2\} \leq \min \left\{P_{pk}, \frac{Q_{pk}}{g_0}\right\} \end{cases}$$ (3.18)

$$P(g_0, g_1, g_2) = 0, \text{or } P(g_0, g_1, g_2) = \min \left\{P_{pk}, \frac{Q_{pk}}{g_0}\right\}$$ (3.19)

In (3.18) we noticed that $F(x)$ (3.6) decreased then increased, and there would be two possible value (3.19) for $P(g_0, g_1, g_2)$ to achieve the maximum result of $F(x)$. In order to find the optimal $P(g_0, g_1, g_2)$, we need to compare both of the results which generated by those two possible $P(g_0, g_1, g_2)$.

If:

$$\begin{cases} \Delta > 0, \\
0 < X_1, X_2 \leq \min \left\{P_{pk}, \frac{Q_{pk}}{g_0}\right\} \end{cases}$$ (3.20)

$$P(g_0, g_1, g_2) = \min \{X_1, X_2\}, \text{or } P(g_0, g_1, g_2) = \min \left\{P_{pk}, \frac{Q_{pk}}{g_0}\right\}$$ (3.21)

In (3.20) we could get two possible value (3.21) as well, and the process to derive the optimal result was similar to (3.18).
If:
\[
\begin{cases}
\Delta > 0 \\
0 < \min \{X_1, X_2\} \leq \min \left\{ P_{pk}, \frac{Q_{pk}}{g_0} \right\} \\
\min \left\{ P_{pk}, \frac{Q_{pk}}{g_0} \right\} < \max \{X_1, X_2\}
\end{cases}
\] (3.22)

\[P(g_0, g_1, g_2) = \min \{X_1, X_2\}\] (3.23)

In (3.22) we noticed that \(F(x)\) first increase then decrease, and the optimal value of \(P(g_0, g_1, g_2)\) would be the \(X_1\) or \(X_2\) which within the range of \((0, \min \left\{ P_{pk}, \frac{Q_{pk}}{g_0} \right\})\).

If
\[
\begin{cases}
\Delta > 0 \\
\min \{X_1, X_2\} \leq 0 \\
\min \left\{ P_{pk}, \frac{Q_{pk}}{g_0} \right\} < \max \{X_1, X_2\}
\end{cases}
\] (3.24)

\[P(g_0, g_1, g_2) = 0\] (3.25)

\(F(x)\) was monotonically decreasing within the range \([0, \min \left\{ P_{pk}, \frac{Q_{pk}}{g_0} \right\}]\). In case (3.24) we could get the optimal value of \(P(g_0, g_1, g_2)\) equal to zero.

### 3.2 Peak transmit power constraint and average interference power constraint

The Lagrangian theory and KKT conditions were considered to solve the optimal function and find out the maximum value of the ergodic capacity. From previous chapter we could observe that the optimal function and its power constraints were shown as follow:

\[
\max_{P(g_0, g_1, g_2) \in \mathbb{F}_2} E[\alpha \log_2(1 + \frac{g_1P(g_0, g_1, g_2)}{N_0}) + (1 - \alpha) \log_2(1 + \frac{g_2P_B}{N_0 + 2g_0P(g_0, g_1, g_2)})] \tag{3.26}
\]

Subject to:
\[
\begin{cases}
P(g_0, g_1, g_2) \geq 0 \\
P(g_0, g_1, g_2) \leq P_{pk} \\
E[\alpha g_0P(g_0, g_1, g_2)] \leq Q_{av}
\end{cases}
\] (3.27)

In order to simplify the function we could make some assumptions which was similar to previous section:
\[
\begin{cases}
P(g_0, g_1, g_2) = 0 \\
\frac{1}{m^2} = k
\end{cases}
\] (3.28)
Based on the Lagrange theory, we could abstract average interference power constraint and get the following equation:

\[
L(x, \lambda) = E\left[\alpha \log_2(1 + \frac{g_1 x}{N_0}) + (1 - \alpha) \log_2(1 + \frac{g_2 P_B}{N_0 + g_0 x})\right] - \lambda (E[g_0 x] - Q_{av}) \tag{3.29}
\]

For a fixed \(\lambda\), the optimal function (3.2) could be simplified as:

\[
\max \left(\alpha \log_2(1 + \frac{g_1 x}{N_0}) + (1 - \alpha) \log_2(1 + \frac{g_2 P_B}{N_0 + g_0 x})\right) - \lambda g_0 x \tag{3.30}
\]

Subject to

\[
0 \leq x \leq P_{pk} \tag{3.31}
\]

According to the KKT conditions, the optimal solution needs to satisfy the following equations:

\[
\begin{cases}
0 \leq x \leq P_{pk} \\
E[g_0 x] \leq Q_{av} \\
\lambda (E[g_0 x] - Q_{av}) = 0
\end{cases} \tag{3.32}
\]

From the KKT conditions (3.32), we noticed that the \(\lambda\) could be either be zero or be derived from \(\lambda (E[g_0 x] - Q_{av}) = 0\). And we could abstract the peak transmit power constraint as well as (3.29) based on the Lagrange theory as well:

\[
L(x, \mu, \nu) = \left(\alpha \log_2(1 + \frac{g_1 x}{N_0}) + (1 - \alpha) \log_2(1 + \frac{g_2 P_B}{N_0 + g_0 x})\right) - \lambda g_0 x - \mu (x - P_{pk}) + \nu x \tag{3.33}
\]

Where the \(\mu\) and \(\nu\) were the nonnegative dual variables associated with the constraints (3.31). Form the KKT condition, the equation (3.33) can show these attributes:

\[
\begin{cases}
\mu (x - P_{pk}) = 0 \\
\nu x = 0 \\
\left[\alpha \log_2\left(1 + \frac{g_1 x}{N_0}\right) + (1 - \alpha) \log_2\left(1 + \frac{g_2 P_B}{N_0 + g_0 x}\right) - \lambda g_0 x - \mu (x - P_{pk}) + \nu x\right]' = 0
\end{cases} \tag{3.34}
\]

Suppose that:

\[
0 < x < P_{pk} \tag{3.35}
\]

And we could get:

\[
\alpha \left(\frac{k g_1}{N_0 + g_1 x}\right) + (1 - \alpha)\left(\frac{-g_0 g_2 P_B k}{(N_0 + g_0 x)^2 + (N_0 + g_0 x) g_2 P_B}\right) - \lambda g_0 = 0 \tag{3.36}
\]

We could obtain that this equation was a cubic function with the variable \(x\). And it has the same structure with basic cubic function which was

\[
a x^3 + b x^2 + c x + d = 0 \tag{3.37}
\]
Where

\[
\begin{align*}
\alpha &= -g_0^3 g_1 \lambda \\
b &= \alpha g_0^2 g_1 k - g_0^2 N_0 \lambda - 2g_0^2 g_1 N_0 \lambda - g_0^2 g_1 g_2 P_B \lambda \\
c &= 2\alpha g_0 g_1 k N_0 - g_0 g_1 N_0^2 \lambda - 2g_0^2 N_0^2 \lambda - g_0 g_1 g_2 k P_B - g_0^2 g_2 N_0 P_B \lambda + 2\alpha g_0 g_1 g_2 k P_B - g_0 g_1 g_2 N_0 P_B \\
d &= \alpha g_1 k N_0^2 - g_0 N_0^2 \lambda - g_0 g_2 k N_0 P_B - g_0 g_2 N_0^2 P_B \lambda + \alpha g_0 g_2 k N_0 P_B + \alpha g_1 g_2 k N_0 P_B
\end{align*}
\]

(3.38)

And we could get these three possible optimal solutions according to the following equations:

\[
\begin{align*}
\Delta &= \left( \frac{b c}{6 a^2} - \frac{b^3}{27 a^3} - \frac{d}{2a} \right)^2 + \left( \frac{c}{3a} - \frac{b^2}{9 a^2} \right)^3 \\
X_1 &= -\frac{b}{3a} + \sqrt[3]{\frac{b c}{6a^2} - \frac{b^3}{27a^3} - \frac{d}{2a} + \sqrt{\Delta}} + \sqrt[3]{\frac{b c}{6a^2} - \frac{b^3}{27a^3} - \frac{d}{2a} - \sqrt{\Delta}} \\
X_2 &= -\frac{b}{3a} + \frac{1}{2} \sqrt[3]{\frac{b c}{6a^2} - \frac{b^3}{27a^3} - \frac{d}{2a} + \sqrt{\Delta}} + \frac{1}{2} \sqrt[3]{\frac{b c}{6a^2} - \frac{b^3}{27a^3} - \frac{d}{2a} - \sqrt{\Delta}} \\
X_3 &= -\frac{b}{3a} + \frac{1}{2} \sqrt[3]{\frac{b c}{6a^2} - \frac{b^3}{27a^3} - \frac{d}{2a} + \sqrt{\Delta}} - \frac{1}{2} \sqrt[3]{\frac{b c}{6a^2} - \frac{b^3}{27a^3} - \frac{d}{2a} - \sqrt{\Delta}}
\end{align*}
\]

(3.39)

Here we need to discuss different situations based on the attributes of the cubic function:

If

\[
\Delta > 0
\]

(3.40)

The equation has one root, which means the optimal \( P(g_0, g_1, g_2) \) equals to the \( X_1 \):

\[
P(g_0, g_1, g_2) = X_1
\]

(3.41)

If

\[
\Delta = \left( \frac{b c}{6a^2} - \frac{b^3}{27a^3} - \frac{d}{2a} \right)^2 + \left( \frac{c}{3a} - \frac{b^2}{9 a^2} \right)^3 = 0
\]

(3.42)

The equation has three equal roots:

\[
P(g_0, g_1, g_2) = X_1 = X_2 = X_3
\]

(3.43)

If

\[
\Delta = \left( \frac{b c}{6a^2} - \frac{b^3}{27a^3} - \frac{d}{2a} \right)^2 + \left( \frac{c}{3a} - \frac{b^2}{9 a^2} \right)^3 \neq 0
\]

(3.44)

The equation has 2 different roots, so the \( P(g_0, g_1, g_2) \) could have two possible values. In order to find the optimal solution of \( P(g_0, g_1, g_2) \), we could compare these two solutions by using the optimal equation (3.2) if

\[
\Delta < 0
\]

(3.45)

The equation has 3 different roots, which means the \( P(g_0, g_1, g_2) \) could have three possible values. And the process to find the optimal solution of \( P(g_0, g_1, g_2) \) was the similar to the previous situation.

13
3.3 Average transmit power constraint and peak interference power constraint

\[
\max_{P(g_0, g_1, g_2) \in F_3} E[\alpha \log_2(1 + \frac{g_1 P(g_0, g_1, g_2)}{N_0}) + (1 - \alpha) \log_2(1 + \frac{g_2 P_B}{N_0 + g_0 P(g_0, g_1, g_2)})]
\] (3.46)

Subject to:

\[
\begin{align*}
P(g_0, g_1, g_2) & \geq 0 \\
E[P(g_0, g_1, g_2)] & \leq P_a \\
P(g_0, g_1, g_2) & \leq g_0 Q_{pk}
\end{align*}
\] (3.47)

Base on the Lagrange theory we could abstract average interference power constraint and get the following equation:

\[
L(x, \lambda) = E[\alpha \log_2(1 + \frac{q_1 x}{N_0}) + (1 - \alpha) \log_2(1 + \frac{g_2 P_B}{N_0 + g_0 x})] - \lambda(x - \frac{g_0 Q_{pk}}{N_0 + g_0 x})
\] (3.48)

For a particular fading state the optimal function could be simplified as:

\[
\max \left( \alpha \log_2(1 + \frac{q_1 x}{N_0}) + (1 - \alpha) \log_2(1 + \frac{g_2 P_B}{N_0 + g_0 x}) \right) - \lambda x
\] (3.49)

And we could abstract the peak transmit power constraint based on the Lagrange theory as well:

\[
L(x, \mu, \nu) = \left( \alpha \log_2(1 + \frac{q_1 x}{N_0}) + (1 - \alpha) \log_2(1 + \frac{g_2 P_B}{N_0 + g_0 x}) \right) - \lambda g_0 x - \mu(x - g_0 Q_{pk}) + \nu
\] (3.50)

Form the KKT condition this equation can show these attributes:

\[
\begin{align*}
\mu(x - g_0 Q_{pk}) &= 0 \\
\nu x &= 0 \\
\left[ \alpha \log_2(1 + \frac{q_1 x}{N_0}) + (1 - \alpha) \log_2(1 + \frac{g_2 P_B}{N_0 + g_0 x}) - \lambda x - \mu(x - g_0 Q_{pk}) + \nu x \right]^T &= 0
\end{align*}
\] (3.51)

From previous functions we could get:

\[
\alpha \left( \frac{kg_1}{N_0 + g_1 x} \right) + (1 - \alpha) \left( \frac{g_2 P_B}{(N_0 + g_0 x)^2 + (N_0 + g_0 x) g_2 P_B} \right) - \lambda = 0
\] (3.52)

We could also solve the problem by using the attributes of (3.2):

\[
\begin{align*}
a &= -g_0^2 g_1 \\
b &= \alpha g_0^2 g_1 k - g_0^2 N_0 \lambda - 2 g_0^3 g_1 N_0 \lambda - g_0^2 g_1 g_2 P_B \lambda \\
c &= 2 g_0 g_1 k N_0 - g_0 g_1 N_0^2 \lambda - 2 g_0^2 N_0^2 \lambda - g_0 g_1 g_2 k P_B - g_0^2 g_2 N_0 P_B \lambda + 2 \alpha g_0 g_1 g_2 k P_B - g_0 g_1 g_2 N_0 P_B \lambda \\
d &= \alpha g_1 k N_0^2 - g_0 N_0^3 \lambda - g_0 g_2 k N_0 P_B - g_0 g_2 n^2 P_B \lambda + \alpha g_0 g_2 k N_0 P_B + \alpha g_1 g_2 k N_0 P_B
\end{align*}
\] (3.53)

And we could get these three possible optimal solutions according to the following equations:

\[
\begin{align*}
\Delta &= \left( \frac{bc}{2a} - \frac{b^3}{2a^2} - \frac{d}{2a} \sqrt{\Delta} + \left( \frac{c}{2a} - \frac{b^2}{2a^2} \right)^2 \right) \sqrt{\Delta}
\end{align*}
\] (3.54)

\[
\begin{align*}
X_1 &= \frac{b}{3a} + \frac{1}{2} \sqrt{\frac{bc}{2a} - \frac{b^3}{2a^2} - \frac{d}{2a} \sqrt{\Delta} + \left( \frac{c}{2a} - \frac{b^2}{2a^2} \right)^2} - \sqrt{\frac{bc}{2a} - \frac{b^3}{2a^2} - \frac{d}{2a} - \sqrt{\Delta}} \\
X_2 &= \frac{b}{3a} + \frac{1}{2} \sqrt{\frac{bc}{2a} - \frac{b^3}{2a^2} - \frac{d}{2a} - \sqrt{\Delta} + \left( \frac{c}{2a} - \frac{b^2}{2a^2} \right)^2} - \sqrt{\frac{bc}{2a} - \frac{b^3}{2a^2} - \frac{d}{2a} - \sqrt{\Delta}} \\
X_3 &= \frac{b}{3a} + \frac{1}{2} \sqrt{\frac{bc}{2a} - \frac{b^3}{2a^2} - \frac{d}{2a} - \sqrt{\Delta} + \left( \frac{c}{2a} - \frac{b^2}{2a^2} \right)^2} - \sqrt{\frac{bc}{2a} - \frac{b^3}{2a^2} - \frac{d}{2a} - \sqrt{\Delta}}
\end{align*}
\]
And the discussions were the same as those in section 3.2

### 3.4 Average transmit power constraint and average interference power constraint

\[
\max_{P(g_0, g_1, g_2) \in P_4} E[\alpha \log_2(1 + \frac{g_1 P(g_0, g_1, g_2)}{N_0}) + (1 - \alpha) \log_2(1 + \frac{g_2 P_B}{N_0 + g_0 P(g_0, g_1, g_2)})] \tag{3.55}
\]

Subject to:

\[
\begin{align*}
& P(g_0, g_1, g_2) \geq 0 \\
& E[P(g_0, g_1, g_2)] \leq P_{av} \\
& E[g_0 P(g_0, g_1, g_2)] \leq Q_{av}
\end{align*}
\]

Base on the Lagrange theory we could abstract average interference power constraint and get the following equation:

\[
L(x, \lambda, \mu) = E\left[\alpha \log_2(1 + \frac{g_1 x}{N_0}) + (1 - \alpha) \log_2(1 + \frac{g_2 P_B}{N_0 + g_0 x})\right] - \lambda(E[x] - P_{av}) - \mu(E[g_0 x] - Q_{av}) \tag{3.56}
\]

Form the KKT condition this equation can show these attributes:

\[
\begin{cases}
\lambda(E[x] - P_{av}) = 0 \\
\mu(E[g_0 x] - Q_{av}) = 0 \\
\left[\alpha \log_2(1 + \frac{g_1 x}{N_0}) + (1 - \alpha) \log_2(1 + \frac{g_2 P_B}{N_0 + g_0 x}) - \lambda x - \mu g_0 x - Q_{av}\right]' = 0
\end{cases} \tag{3.58}
\]

Then the equation would be expressed as:

\[
\alpha \left(\frac{k g_1}{N_0 + g_0 x}\right) + (1 - \alpha) \left(\frac{-g_0 g_2 P_B k}{(N_0 + g_0 x)^2 + (N_0 + g_0 x) g_2 P_B}\right) - \lambda - \mu = 0 \tag{3.59}
\]

We could solve the problem by using the attributes of (3.2) as well, and the discussion processes were the same as the previous section:

\[
\begin{align*}
& a = -g_0^2 \\
& b = \alpha g_1 g_1 k - g_0^2 N_0 \lambda - 2 g_0 g_1 N_0 \mu - g_0 g_1 g_2 P_B \lambda \\
& c = 2 \alpha g_0 g_1 k N_0 - g_0 g_1 N_0^2 \lambda - 2 g_0^2 N_0^2 \mu - g_0 g_1 g_2 k P_B - g_0^2 g_2 N_0 P_B \lambda - g_0 g_1 g_2 N_0 P_B \mu \\
& d = \alpha g_1 k N_0^2 - g_0 N_0^2 \lambda + \alpha g_0 g_2 k N_0 P_B + \alpha g_1 k N_0 P_B
\end{align*} \tag{3.60}
\]

And we could get these three possible optimal solutions according to the following equations:

\[
\begin{align*}
\Delta &= \left(\frac{b c}{6 a^2} - \frac{b^3}{27 a^3} - \frac{d}{2 a}\right)^2 + \left(\frac{c}{3 a} - \frac{b^2}{9 a^2}\right)^3 \\
X_1 &= -\frac{b}{3 a} + \sqrt[3]{\frac{b c}{6 a^2} - \frac{b^3}{27 a^3} - \frac{d}{2 a} + \sqrt{\Delta}} + \sqrt[3]{\frac{b c}{6 a^2} - \frac{b^3}{27 a^3} - \frac{d}{2 a} - \sqrt{\Delta}} \\
X_2 &= -\frac{b}{3 a} + \frac{1}{2} \sqrt[3]{\frac{b c}{6 a^2} - \frac{b^3}{27 a^3} - \frac{d}{2 a} + \sqrt{\Delta}} - \frac{1}{2} \sqrt[3]{\frac{b c}{6 a^2} - \frac{b^3}{27 a^3} - \frac{d}{2 a} - \sqrt{\Delta}} \\
X_3 &= -\frac{b}{3 a} + \frac{1}{2} \sqrt[3]{\frac{b c}{6 a^2} - \frac{b^3}{27 a^3} - \frac{d}{2 a} + \sqrt{\Delta}} - \frac{1}{2} \sqrt[3]{\frac{b c}{6 a^2} - \frac{b^3}{27 a^3} - \frac{d}{2 a} - \sqrt{\Delta}} \tag{3.61}
\end{align*}
\]
Chapter 4

Numerical Results

In this chapter, the simulation results for ergodic capacity are presented. For Rayleigh fading channels, the channel power gains (exponentially distributed) are assumed to be of unit mean. For AWGN channels, the noise power is also assumed to be one.

4.1 Simulation results

Figure 4.1: Peak transmit and peak interference power constraints.

Figure 4.1 shows the ergodic capacity of the whole system under the constraints of peak transmit power and peak interference power for the weight ranged from 0 to 1. When \( a \) was increased which means the weight of system power allocation for SU network was raising but the system power allocated for PU network was decreased. It was observed that \( P(g_0, g_1, g_2) \) would be the maximum value when \( P(g_0, g_1, g_2) \) was the right point of its constraint range, i.e.,
$P(g_0, g_1, g_2)$ equals to the minimum value between $P_{pk}$ and $Q_{pk}/g_0$. Assume the x axis was the $P(g_0, g_1, g_2)$, y axis was the ergodic capacity. From this simulation result, we could have following conclusions. When $\alpha = 0$, it means the PU was only considered. The ergodic capacity was decreasing while $P(g_0, g_1, g_2)$ was increasing, since $P(g_0, g_1, g_2)$ was the noise to PU. When $\alpha = 1$, it means the SU was only considered. The curve shows that the ergodic capacity was rising while $P(g_0, g_1, g_2)$ was increasing, since $P(g_0, g_1, g_2)$ was the signal power of SU. When $\alpha = 0.5$, it means the power allocations between PU and SU were equal. The result shows a slow growth of the ergodic capacity. We concluded that the increasing of $P(g_0, g_1, g_2)$ has more affection on the rate increase of SU than on the rate decay of PU and leads to increase the whole system ergodic capacity. A noticeable result shown in Figure 4.1 was all the curves interest at one point. It indicated that SU and PU have equal rates and the weight value of $\alpha$ does not make sense when $P(g_0, g_1, g_2)$ is equal to the specific value.

Figure 4.2: Peak transmit and average interference power constraints with variable $P_B$.

Figure 4.2 shows the ergodic capacity of whole system against $Q_{av}$ ranged from $-20dBm$ to $20dBm$ under peak transmit and average interference power constraints. It shows that the ergodic capacity was increasing with an increasing value of average interference constraint $Q_{av}$. That is because the increase of $Q_{av}$ would lead to an increase of the SU rate while has a minor impact on the degradation of PU rate. It is also interesting to note that the ergodic capacity was increasing with a slow rate when $Q_{av}$ was small, and vice versa. Given a $Q_{av}$ value, a larger $P_B$ would have a larger ergodic capacity. That is because the PU rate was increased. We can also observe that the ergodic capacity under a small $Q_{av}$ was increased significantly with an increasing $P_B$ while the ergodic capacity under a large $Q_{av}$ did not show much improvement. We could conclude that with a small $Q_{av}$, the increasing $P_B$ has a significant positive effect to
the ergodic capacity. In a large $Q_{av}$ case, such an affection from $P_B$ was reduced.

Figure 4.3: Peak transmit and average interference power constraints with variable $\alpha$.

Figure 4.3 shows the ergodic capacity versus the weight value $\alpha$ under peak transmit and average interference power constraints. For $P_B = 20dBm$, $Q_{av} = 20dBm$, we could observe the ergodic capacity would first increase then decrease with the increasing weight value. For the cases of $P_B = 10dBm$, $Q_{av} = 10dBm$ and of $P_B = 0dBm$, $Q_{av} = 0dBm$, those two curves have the similar tendency with first curve. However, since we had the power constraint $P(g_0, g_1, g_2) \leq 0$, the ergodic capacity does not exist for those $\alpha$ which result in negative optimal $P(g_0, g_1, g_2)$ in our optimize function.

Figure 4.4 also shows the ergodic capacity versus the weight value $\alpha$ under peak transmit and average interference power constraints. Given $P_B = 20dBm$ and $Q_{av}$ with different values of 1dBm, 10dBm, 20dBm, we could observe that the corresponding curves were quite different from each other. That is because the parameters $P_B$, $Q_{av}$, and $\alpha$ were both involved during the optimization process. The output objective ergodic capacity here is a complicated function of $\alpha$ whose coefficients are determined by the fixed values of $P_B$ and $Q_{av}$. Different values of $P_B$ and $Q_{av}$ lead to objective functions of quite different properties, e.g., different feasible sets of $\alpha$ and increasing/decreasing property.

Figure 4.5 shows the ergodic capacity versus $P_{av}$ changed from $-20dBm$ to $20dBm$ under the average transmit power and peak interference power constraints. We simulated three curves of $P_B = 0dBm$, $P_B = 10dBm$, and $P_B = 20dBm$ respectively. Obviously, Figure 4.5 has the same property as Figure 4.2. The reason was our assumption of $g_0 = g_1 = g_2 = 1$ in the simulation. Assume the $\lambda$ in $F_2$ scenario was $\lambda_1$ and $\lambda$ in $F_3$ scenario was $\lambda_2$. Based
Figure 4.4: Peak transmit and average interference power constraints with variable $\alpha$.

Figure 4.5: Average transmit and peak interference power constraints with variable $P_B$. 

19
on our optimization process $\lambda_1$ could be obtained through following equation:

$$E[g_0P(g_0, g_1, g_2)] = Q_{av}$$

And 2 could be obtained through following equation:

$$E[P(g_0, g_1, g_2)] = P_{av}$$

Since $g_0 = 1$, the numerical value of $\lambda_1$ was equal to $\lambda_2$ if $Q_{av}$ has the same value with $P_{av}$. Regarding those explanation we could draw the same conclusion with Figure 4.2.

![Figure 4.6: Average transmit and Average interference power constraints with variable $P_{av}$.](image)

Figure 4.6 shows the ergodic capacity versus $P_{av}$ under the average transmit and average interference power constraints. Here, set that $\alpha = 0.5$ and $P_{av}$ is ranged from $10dBm$ to $20dBm$. These three curves were simulated based on different values of $Q_{av}$, which were $Q_{av} = 0dBm$, $Q_{av} = 1dBm$, or $Q_{av} = 20dBm$. From the simulation result, we could observe that the ergodic capacity was rising when $P_{av}$ was increasing. When $Q_{av}$ was small the increasing rate was large, and the increasing rate was reduce if $Q_{av}$ was large which the curve would be more flat. The value of $Q_{av}$ could also affect the ergodic capacity value with the initial point of $P_{av}$, larger $Q_{av}$ would lead to more large initial ergodic capacity.

### 4.2 Summary

In this chapter, the numerical result was presented. The ergodic capacity under four combinations of power constraints were discussed respectively; the variables $\alpha$, $Q_{pk}$, $P_{pk}$, $Q_{av}$, $P_{av}$, and $P_B$ and their affections to ergodic capacity of the
whole system had been discussed as well. The explanation of the operational meanings related to those numerical result had be analyzed. The optimal solutions and the reasons were also illustrated based on the optimal functions which described in previous chapter.
Chapter 5

Conclusion

In this paper, the optimal power allocation strategies to achieve the ergodic capacities of a SU fading channel under spectrum sharing are studied, subject to different combinations of peak/average transmit and peak/average interference power constraints. It is shown that under the same transmit and interference power constraints, different weight value $\alpha$ would affect the properties of optimal function refer to the non-liner variety. As for different average or peak transmit/interference power constraints they were shown the same numerical results, however the results related to the variables assumption for channel power gain $(g_0, g_1, g_2)$. Which means different groups of $(g_0, g_1, g_2)$ would result in different optimal solutions, which shown in the chapter 3. This paper only illustrate the common result of the optimal power allocation solution based on our assumptions in chapter 4, but more study on the affections of different channel power gains to the optimal results were need to be considered indeed.


