Study on Vibrations induced by Parametric Excitations on Strings

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Abstract:
The present thesis work studies the parametrically excited vibrations on bridge stay cables when the natural frequency of the bridge tower is half of the resonance frequency of the cable. This condition is found at the longest stay cables of the Øresund Bridge. Experimental and numerical analysis is carried out based on a simplified scaled model (1:200) of this bridge, replacing the cables by strings made of steel. When parametrically excited vibrations are studied, certain relations between resonance frequencies of bridge tower and cable lead to high vibration levels. The most critical relation is when the tower frequency is double the cable frequency. The case studied here, where the tower frequency is half of the cable frequency, shows to be less severe.

Keywords:
Vibrations, Parametric Excitations, Strings, Stay Cables, Modal Analysis.
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Karlskrona, April 2007.

Armando Enrique León Guarena.
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1 Notation

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<tr>
<td>$A$</td>
<td>Cross Sectional Area [m$^2$]</td>
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<td>$c$</td>
<td>Viscous damping [Ns/m]</td>
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<td>$E$</td>
<td>Modulus of Elasticity [N/m$^2$]</td>
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<tr>
<td>$z$</td>
<td>Cable Stretching [m]</td>
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<tr>
<td>$\delta$</td>
<td>Cable Elastic Extension [m]</td>
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</table>
\( \omega \) Vibration Resonance [rad/s]

\( \rho \) Density [kg/m\(^3\)]

\( \theta \) Angle [rad], [degrees]

\( \chi \) Cable Tautness

\( \psi \) Shape Mode Function

\( \zeta \) Relative damping coefficient

**Indices**

\( \theta \) Initial, Static, Natural Undamped

\( d \) Dynamic, Damped.

\( c \) Cable

\( p \) Pylon

**Abbreviations**

FEM Finite Element Method

FRF Frequency Response Function

MDOF Multi Degree of Freedom

RMS Root Mean Square

SDOF Single Degree of Freedom
2. Introduction

In Structural Engineering, cable-stayed bridges have become very popular and useful due to their attractive aesthetics and their effective construction process. During its design period, the need for longer span distances has been considered as a challenge to face, since that means the use of longer stay cables, which consequently are more vulnerable to vibrations induced by seismic motions, wind-rain excitations and support motions.

This thesis report deals with an important source of vibration, which has been found in the support motions and defined as Parametric Excitations. Those are characterized by periodic displacements at the tower and/or deck of the bridge, inducing tension oscillations at the cables which can be accompanied by large amplitude vibrations.

The aim of the present work is to study by experimental and numerical analysis the Parametrically Excited Vibrations on bridge-stay cables, taking as a reference a real case, the Øresund Bridge. The study is developed on a simplified scaled model (1:200) of this bridge, and it is focus on its longest cables, where the larger vibrations amplitudes have been reported.

Initially, some theoretical fundamentals about vibration, modal analysis and parametric excitations are presented in chapter 3, while some dynamic models of stay cables are described in chapter 4. Then, the methodology employed to carry out the numerical and experimental analysis is presented in chapter 5, and consequently in the following chapters are presented the results, discussion and the conclusion of the research work.

2.1 Background

Different methodologies based on analytical, numerical and experimental results have been applied in order to study the stay cables dynamics under parametric excitations over the last 20 years by different researchers. Originally, the mathematical background of parametric excited vibrations remains on the application of the Mathieu Equation [1] (presented at the end of the 19th century), which basically looks for the determination of those regions in which the studied system gets an unstable behavior characterized by large amplitudes.
However, the dynamic of the stay cables has implied the introduction of particular numerical models which are looking for better approximations. In 1994, Pinto and Lilien [2] presented a numerical analysis based on a model which described the stay cable as a continuous system with one of its anchorages free to move periodically, in order to reproduce the support motions. In that model, although the conservative results when considering large stay cables (a 15 m. peak to peak transverse amplitude for a 440 m stay cable), maximum amplitudes were obtained when the frequency of the free anchorage was about one and two times the first-natural resonance frequency of the cable. Usually this relationship between resonance frequencies is described by a ratio in which the first natural resonance frequency of the cable is taken as a reference; then, the mentioned cases will be identified by a frequency ratio equal to 1 and 2, respectively.

The interaction between the tower and/or deck of the bridge with the cable has been also considered by other researchers. Sun et al. [3] developed an integrated model considering the behavior of the cable and tower/deck as depending on each other. In that research, a numerical analysis was carried out taking into account the first natural resonance frequency of the bridge tower/deck double the cable-resonance frequency (ratio=2), since that condition is usually referred as the most critical one in parametric excitation [2],[3]. The numerical results showed amplitudes for the cable around 1 meter when the deck was initially excited by displacements around 10 cm in amplitude.

Based on the interaction only between deck and cable, Gattulli, Paolone and D’Amore [4] in order to describe the deck motion, modeled this structural bridge component as a cantilever beam with its free tip attached to one end of the cable, while the remaining cable end was kept fixed by an anchorage to ground. Their analysis conducted to observe a non-linear phenomena between cable and deck, which was mainly noticed when the frequency ratio deck-cable was about 0.5 and 2.

Despite most of the research works emphases their analysis on the case in which the excitation frequency is two times the cable-resonance frequency (ratio=2) as the most critical condition in Parametric Excitation on cables since the largest amplitudes are expected under such condition; in some ones [2],[5] the authors have also evaluated by simulations the condition in which the resonance frequency ratio is 1 and even 0.5. As a result, still large
amplitudes were reported when the ratio was 1, but significantly reduced vibration amplitudes were obtained for the case in which the ratio was 0.5, if they are compared to the other two conditions. Although the main focus on punctual cases, the parametric excitation on cables could be expected, in theory according to Pinto and Lilien [2], depending on how large the excitation amplitudes are, regardless the resonance frequency ratio.

It is a fact that cable vibrations have been observed on many cables-stayed bridges around the world, as in the Øresund Bridge, which joins the cities of Malmö at the south of Sweden and Copenhagen, Denmark. This important link for both countries is continuously monitored and large amplitudes about 2.5 and 3 meters at the longest stay cables have been reported, in which parametric excitations could be partially behind these incidences [5]. By far, such amplitudes result quite uncomfortable for the users of the bridge, however, fatigue at the cables and stresses at their anchorages are the structurally aspects to evaluate in order to guarantee the right operation of the bridge.

Therefore, in 2005, a first analysis by Abu-Iseifan and Paulson [6] under the supervision of Prof. Kjell Ahlin, was carried out on a simplified scaled model (1:200) of the Øresund Bridge at the laboratories of Blekinge Tekniska Högskola, BTH. In such scaled model, the cables were replaced by piano strings made of steel. In that case, the bridge tower resonance frequency was about two times the first resonance frequency of the string (ratio=2), in agreement with it has been considered as the most critical condition according to most of the previous researches in Parametric Excitations on cables [2],[3],[5]. As a result, large amplitudes were observed experimentally only under such condition, while no parametric excited vibrations were observed when the ratio between the natural frequencies of the tower and the cable was slightly changed from 2.

However, in reality at the Øresund Bridge, the resonance frequency for the tower is about half of the resonance frequency of the longest stay cable (ratio=0.5). For this reason, the present thesis work evaluate experimentally and numerically the real conditions for the considered bridge, taking as well the previous analysis [2],[5] in which the vibration amplitudes are expected to be smaller under such frequency ratio, meaning a more convenient condition when it is compared to the most critical case recognized when such ratio is 2.
3. Theoretical Fundaments

3.1 Dynamic of the Transverse Vibration of a Taut String

A cable or string in tension is a continuous member whose geometric configuration can be described by the position of its axial line with respect to a fixed coordinate line. As follows is explained the development of the corresponding governing equation for the transverse vibration of a taut string, according to reference [7].

Firstly, in order to study its transverse vibration, which implies its perpendicular displacements with respect to its axis, some assumptions are considered:

- The lateral dimensions are much smaller than the longitudinal one, then, the cable can be considered as a line structure.
- The cable or string remains in a single plane, presenting displacements along its length on the transverse direction.
- The cable tension remains constant during the cable motion.
- Small variations in slope are considered along the cable.

![Figure 3.1 Infinitesimal part of a taut cable.](image)
Considering an infinitesimal part of the cable at its position \( x \), as shown in figure 3.1., the transverse vibration formulation can be obtained by applying the Newton’s Second Law as follows:

\[
f(x, t) - T \sin \theta + T \sin(\theta + d\theta) = m(x)dx \frac{\partial^2 v(x, t)}{\partial t^2}
\]

(3.1)

where:

- \( f(x, t) \) is the external transverse force per unit of length acting on the cable.
- \( v(x, t) \) is the transverse displacement of the cable, depending on position and time.
- \( T \) is the constant tension along the cable.
- \( \theta \) is the slope of the cable at the position \( x \).
- \( m(x) \) is the cable mass per unit of length.

Since small slope is assumed, \( \sin \theta \) and \( \sin(\theta + d\theta) \) can be approximated to \( \theta \) and \( (\theta + d\theta) \), respectively. Additionally, by trigonometry we have \( \theta = \frac{\partial v}{\partial x} \), and consequently \( d\theta = \frac{\partial^2 v}{\partial x^2} dx \).

Then, in order to obtain the corresponding governing equation for the transverse vibration of a taut cable, the Eq.(3.1) can be rewritten as follows:

\[
m(x) \frac{\partial^2 v(x, t)}{\partial t^2} = T \frac{\partial^2 v(x, t)}{\partial x^2} + f(x, t)
\]

(3.2)

The Eq.(3.2) is a partial differential equation, which can be solved by using the method of separation of variables [8]. As a consequence, \( v(x, t) \) is expressed as the multiplication of one function depending uniquely on the space with one function depending only on time, as follows:

\[
v(x, t) = Y(x)Q(t)
\]

(3.3)

Introducing Eq.(3.3) into Eq.(3.2) and considering the free vibration case (without external force), give us:
\( Y(x) \frac{d^2 Q(t)}{dt^2} = c^2 \frac{d^2 Y(x)}{dx^2} Q(t) \) \hspace{1cm} (3.4)

where \( c = \sqrt{\frac{T}{m}} \), when \( m \), which is the cable mass per unit of length, remains constant along the \( x \)-axis.

Rearranging Eq.(3.4) leads to:

\[
\frac{Y'(x)}{Y(x)} = \frac{1}{c^2} \frac{\ddot{Q}(t)}{Q(t)} = -\lambda^2
\] \hspace{1cm} (3.5)

In Eq.(3.5), there are two fractions at two sides of the equality: one expressed as function depending uniquely on space and another one depending on time. To validate such equality, both of those fractions must be equals to a constant, which is chosen to be negative in order to obtain an oscillatory solution for \( Q(t) \) and non trivial solution for \( Y(x) \).

From Eq.(3.5), two differential equations can be obtained, as follows:

\[
Y''(x) + \lambda^2 Y(x) = 0
\] \hspace{1cm} (3.6)

\[
\ddot{Q}(t) + (\lambda c)^2 Q(t) = 0
\] \hspace{1cm} (3.7)

The general solution for Eq.(3.6) is given by:

\[
Y(x) = A \cos(\lambda x) + B \sin(\lambda x)
\] \hspace{1cm} (3.8)

where \( A \) and \( B \) are two constants. To solve Eq.(3.8) are need the two corresponding boundary conditions, which for a taut cable with its both ends fixed are as follows:

\[
Y(0) = Y(l) = 0
\] \hspace{1cm} (3.9)

By evaluating the Eq.(3.8) with the boundary conditions expressed by (3.9), where, \( l \), represents the length of the cable, we obtain that:
\[ A = 0 \]
\[ B \sin(\lambda l) = 0 \]  \hspace{1cm} (3.10)

From (3.10) an infinite number of solutions are obtained for \( \lambda \):

\[ \lambda_n = \frac{n\pi}{l}, \ n=1,2,3,\ldots, \infty. \]  \hspace{1cm} (3.11)

Then, \( Y(x) \) can finally be expressed as:

\[ Y_n(x) = B_n \sin \left( \frac{n\pi}{l} x \right), \ n=1,2,3,\ldots, \infty. \]  \hspace{1cm} (3.12)

\( Y_n(x) \) represents the different mode shape functions for the cable, describing the infinite number of shapes that the cable can take when vibrating. The figure 3.2 shows the first three natural mode shapes for a taut cable whose ends are kept fixed.

\[ \text{Figure 3.2 First three natural mode shapes for a taut cable with its both ends fixed.} \]
On the other hand, solving the Eq.(3.7) by using the result obtained for $\lambda$ at (3.11), gives for $Q(t)$ the following:

$$Q_n(t) = C_n \sin(\omega_n t + \phi_n)$$  \hspace{1cm} (3.13)

where $C_n$ and $\phi_n$ are two constants depending on the initial conditions (time conditions), while $\omega_n$, which are called the respective natural frequencies of vibration, are given by:

$$\omega_n = \lambda_n c = \frac{n\pi}{l} \sqrt{\frac{T}{m}}, \ n=1,2,3,\ldots, \infty.$$  \hspace{1cm} (3.14)

$Q_n(t)$, is usually called the generalized coordinate and represents the dependency on time of the transverse vibration for the cable; while $\omega_n$ are also called the harmonic resonances of the cable.

As a consequence, the solution for $v(x,t)$, according to the method of separation of variables, will be determined by an infinite summation of the products of the corresponding mode shape functions with the respective generalized coordinate as follows:

$$v(x,t) = \sum C_n \sin \left( \frac{n\pi}{l} x \sin(\omega_n t + \phi_n) \right)$$  \hspace{1cm} (3.15)

It is noticed that in Eq.(3.15) the constant coming from the mode shape functions has been absorbed by the constant $C_n$ from the generalized coordinate equation.

It can be demonstrated that the mode shape functions given by $Y_n(x)$, are independent from each other, as follows:

$$\int_0^l \sin \left( \frac{i\pi}{l} x \right) \sin \left( \frac{j\pi}{l} x \right) dx = 0, \ \text{for} \ i \neq j$$

and

$$\int_0^l \sin \left( \frac{i\pi}{l} x \right) \sin \left( \frac{j\pi}{l} x \right) dx = \frac{l}{2}, \ \text{for} \ i = j$$  \hspace{1cm} (3.16)
Such independence between mode shapes is called the principle of orthogonality. That means, for any natural frequency, $\omega_n$, the cable will keep an unique associated mode shape while vibrating at such frequency. This is the basic for what is known as modal analysis, in which the total displacement of any continuous system, like a beam, rod, cable, etc., can be obtained as the superposition or summation of each mode shape multiplied by its corresponding generalized coordinate.

3.2 Dynamics of Lumped Mass Systems

In general, mechanical systems consist of structural elements which have distributed mass and elasticity, like for example a beam, rod or cable or structures compound by several of them. In many cases, they can be represented by an equivalent more simplified system consisting of lumped solids which are treated as rigid elements with specified inertia effects connected by elastic elements, like springs, which have negligible inertia effects [9].

The easiest way to study the vibration of any system is by obtaining its Single Degree of Freedom (SDOF) model. A SDOF system consist of a lumped mass connected to ground by an elastic element as a spring and by a viscous damper, as shown in figure 3.3, where $F$ represents an external force, $m$ the mass of the system, $k$ the stiffness and $c$ the damping.

![Figure 3.3 Representation of a SDOF system.](image_url)
In that case, the SDOF would describe the displacement or vibration through one representative point within the corresponding mechanical system, like for example its center of mass, the end point of a cantilever beam subjected to an external force, the middle point of a cable, etc, based on an equivalent estimated mass, stiffness and damping from the mechanical properties of the system.

By applying the Newton’s Second Law, the governing differential equation for a SDOF is represented by:

\[ m \ddot{x} + c \dot{x} + kx = F \]  

\[(3.17)\]

Solving Eq.(3.17) when the external force is zero by using the Euler’s method [8], give us the solution for a free vibration case, whose expression is given by:

\[ x(t) = Ae^{-ζω_d t} \sin(ω_d t + φ) \]  

\[(3.18)\]

where A and φ are two constants determined by evaluating the two corresponding initial conditions; while ζ, ω₀ and ω_d are respectively called the relative damping, the undamped natural frequency and the damped frequency of the system, given by:

\[ ζ = \frac{c}{2\sqrt{km}} \]  

\[(3.19)\]

\[ ω₀ = \sqrt{\frac{k}{m}} \]  

\[(3.20)\]

\[ ω_d = ω₀ \sqrt{1 - ζ^2} \]  

\[(3.21)\]

According to (3.18) the response on time domain for the SDOF system will be sinusoidal, which is expected for oscillatory motions, with a ω_d as frequency and will be attenuated by the relative damping, ζ, until the displacement completely is vanished, as shown in figure 3.4.
Solving Eq.(3.17) by applying the Laplace Transform [10], leads to obtain what is called the transfer function $H(s)$ [11], which describe in the Laplace domain the relationship between the force and the displacement.

$$H(s) = \frac{X(s)}{F(s)} = \frac{1/m}{s^2 + 2\zeta\omega_0 s + \omega_0^2} \quad \text{(3.22)}$$

By replacing the Laplace variable, $s$, as $s = j2\pi f$, where $j$ indicates the imaginary unit, we can express the transfer function, $H(s)$, on the frequency domain, which is very useful when studying vibrations by experimental measurements, since in that way the peak of the transfer function is found at the vibration frequency of the system, as well as allowing to determine the characteristic properties of the system, as the mass, the damping and the stiffness by applying methodologies which will be described later on.

Figure 3.5 shows an example of a Transfer Function as a function of the frequency, corresponding to a system which is vibrating at around 5 Hz. In vibrations theory, the natural frequency of vibration of the system is usually called resonance frequency, which is defined as the frequency where any system reaches its maximum vibration amplitude.
However, the displacement for mechanical systems, in some cases, can not be just described by obtaining a SDOF model. Depending on how many degrees of freedom are mainly needed to be considered to properly describe its displacement, the lumped mass model will incorporate as many masses as degrees of freedom are taken into account for the model, connected by their respective springs and dampers. In that case, the model is called a multi-degree of freedom (MDOF) system.

In figure 3.6 is shown a system with two degrees of freedom, whose motion will be described by a system of two differential equations as follows:

\[
\begin{align*}
    m_1 \ddot{x}_1 + (c_1 + c_2) \dot{x}_1 + (k_1 + k_2)x_1 - c_2 \dot{x}_2 - k_2x_2 &= F_1(t) \\
    m_2 \ddot{x}_2 + (c_2 + c_3) \dot{x}_2 + (k_2 + k_3)x_2 - c_2 \dot{x}_1 - k_2x_1 &= F_2(t)
\end{align*}
\]

(3.23)
Figure 3.6 Representation of a MDOF system.

For the previous system described by (3.23), $x_1$ and $x_2$ represent the displacements of the masses $m_1$ and $m_2$, respectively, from their equilibrium positions.

When considering the free vibration case with no damping, the solution leads to find the corresponding eigenvalues and eigenvectors. Then, the free vibration and undamped solution for the previous system is expressed as follows [12]:

$$
\begin{bmatrix}
  x_1 \\
  x_2
\end{bmatrix} =
\begin{bmatrix}
  A_{11} \\
  A_{21}
\end{bmatrix} \sin(\omega_1 t + \phi_1) +
\begin{bmatrix}
  A_{12} \\
  A_{22}
\end{bmatrix} \sin(\omega_2 t + \phi_2)
$$

(3.24)

where the $A$'s (amplitudes) and $\phi$'s (phases) are constants determined by the initial conditions.

In Eq.(3.24) the subscripts for the $A$'s mean, firstly the coordinate; while for the second one means at which frequency. For example, $A_{12}$ means the amplitude for the coordinate $x_1$ corresponding to the first mass, at the frequency $\omega_2$. From the eigenvectors, relatives values for the amplitudes $A$'s, can be determined, and from the eigenvalues, the frequencies $\omega_1$ and $\omega_2$.

Consequently, by observing Eq.(3.24) can be noticed that the displacement for each mass will be the result of the contribution given by each degree of freedom, as a summation of the sinusoidal functions at frequencies $\omega_1$ and $\omega_2$. 
\( \omega_2 \), multiplied by their corresponding amplitudes \((A's)\). This is graphically shown in figure 3.7 for the displacement of the mass \( m_1 \).

![Figure 3.7 Displacement as superposition of modes affected by its respective amplitudes.](image)

### 3.3 Modal Analysis and Extraction of the Modal Parameters

Modal Analysis is an important tool in vibration analysis, which is based on the fact that mechanical systems can be described as models that have preferred frequencies and geometric configurations (natural mode shapes) in which they tend to execute free vibration [7].

As seen for the transverse vibration of a continuous system like a cable, the vibrations can be described as the contribution or superposition of their mode shapes multiplied by their respective amplitudes or generalized coordinates. This fact, leads to express the corresponding Transfer function (relationship usually between displacement and force) in the Laplace domain as follows:

\[
H_{ik}(s) = \sum_{r=1}^{n} \frac{(\psi^r_j \psi^r_k)}{s^2 + 2\zeta_j \omega_j s + \omega_j^2} \tag{3.25}
\]

In Eq.(3.25), the subscripts indicate the response point and excitation point on a continuous or in a lumped mass system. For example, for the MDOF system studied before, \( H_{12} \) will indicate the transfer function when the response is measured at the mass \( m_1 \), while only one external force was acting on the mass \( m_2 \). In the case of a continuous structure like a beam, the subscripts will indicate just the points where the structure is measured and
excited, respectively. It is also noticed the need of obtaining certain number of transfer functions in order to characterize the vibration of any system, depending on its number of degree of freedom, and its geometry in the case of a continuous system.

In fact, in practice, when studying any system, which is in reality a continuous one, several transfer functions are experimentally obtained by exciting and measuring the response on convenient points along the structure. Then, each main mode shape of the structure can be obtained and consequently, the vibration of the whole structure can be properly represented.

Considering again the Eq. (3.25), the numerator which is called the residue is associated to the mode shapes, \( \psi \); while from the denominator are determined the poles which are expressed in terms of the frequency and its respective damping from each of those shapes. In simple words, the transfer function will be a summation of partial fractions represented by their residues and poles, which consequently provide the shape modes, frequencies of vibration and damping. In fact, Eq.(3.25) can be rewritten as [13]:

\[
H_{ik} = \sum_{r=1}^{N} \frac{R_r}{s - s_r} + \frac{R_r^*}{s - s_r^*}
\]

(3.26)

where: \( s_r \) and \( R_r \) represent the pole and residue corresponding to the \( r \)-th mode shape, respectively. The symbol * is used to denote the complex conjugate for the pole and residue.

From (3.25) and (3.26) is known that the pole, \( s_r \), is given by:

\[
s_r = -\zeta_r \omega_{0r} + j \omega_{dr}
\]

(3.27)

where \( \zeta_r \), \( \omega_{0r} \) and \( \omega_{dr} \) are the relative damping, the natural undamped frequency and the damped frequency for the \( r \)-th mode, respectively.
Figure 3.8 Frequency Response Function for a two-degree of freedom system.

Figure 3.8 shows an example of a Transfer function expressed in the frequency domain, which is recalled as Frequency Response Function (FRF). Such example could be coming from a two-degree of freedom system (figure 3.6) when the force is exciting one mass and the response is obtained in that mass or on the other one. On the other hand, such FRF could be also from any continuous system which exhibit two main degrees of freedom or mode shapes, when the structure is excited on one point and the response is measured on the same point or on other else, along the structure. Such estimation can be given, since the FRF shows two peaks (around 2.5 Hz and 7 Hz, in this case), which would be related to each resonance frequency of each mode shape or degree of freedom.

Experimentally, FRF’s are quite often obtained and used, since they can show the natural frequencies in which the system used to vibrate, as well as by the fact that allow us to obtain the mode shape, damping and frequencies from their residues and poles, which can be extracted and mathematically determined. Such properties as the mode shape, damping and frequencies are called the modal parameters.

To extract the modal parameters from a FRF, firstly the poles must be determined by evaluating the expression given by Eq.(3.27), after defining
the respective damping, natural frequency, $\omega_{or}$ and the damped frequency, $\omega_{dr}$. Due to the fact that the damping in structures are used to be small ($\zeta << 1$), according to Eq.(3.21), the undamped natural frequency, $\omega_{or}$, can be approximated to the damped frequency $\omega_{dr}$, which are directly recognized from the FRF at their peak values (See for example figure 3.8).

The damping can be estimated by applying the 3 dB-bandwidth method [11] as follows:

- Defining the 3 dB- bandwidth, $B_r$, as:

$$B_r = f_u - f_l$$

with $|H(f_u)|^2 = |H(f_l)|^2 = \frac{1}{2}|H(f_d)|^2$  \hspace{1cm} (3.28)

where $f_u$ and $f_l$ are the upper and lower frequencies found when accomplishing the condition established by (3.28), in which $f_d$ represent the frequency for the FRF-peak value at the considered mode shape or degree of freedom.

- Then, the corresponding damping for each mode shape can be calculated by the following expression:

$$\zeta = \frac{B_r}{f_d}$$  \hspace{1cm} (3.29)

The damping can be also determined by using the Nyquist method [11].

Once the poles are defined, the residues can be estimated by following few steps:

- The Eq.(3.26) can be rewritten by replacing $s=j\omega$ and the poles in the form $s_r = \sigma_r + j\omega_{dr}$, as follows:

$$H_{ik}(j\omega) = \sum_{r=1}^{N} \frac{R_r}{\sigma_r + j(\omega - \omega_r)} + \frac{R_r^*}{\sigma_r + j(\omega + \omega_r)} \hspace{1cm} (3.30)$$
- By selecting the partner points as the peak values and their corresponding frequencies from the graphic of the FRF, and evaluating (3.30) with them, the residues can be obtained through a system of as many equations as degrees of freedom are contained into the FRF, by reducing Eq.(3.30) into:

\[
H_{ik}(j\omega_r) = \frac{R_r}{\sigma_r} + \sum_{n=2}^{N} \frac{R_n}{\sigma_n + j(\omega_r - \omega_n)}
\]  

(3.31)

For example, considering the two-degrees of freedom case represented by figure 3.6, two unknown residues must be estimated (one for each degree). Once, their two poles have been calculated as was mentioned above, the peak values and the respective frequencies can be picked up from the FRF-graphic. Then, when Eq.(3.30) is evaluated at each peak value, for instance at its first frequency, we obtain an expression in the form of Eq.(3.31) as follows:

\[
H_{ik}(j\omega_1) = \frac{R_1}{\sigma_1} + \frac{R_2}{\sigma_2 + j(\omega_1 - \omega_2)}
\]  

(3.32)

In Eq.(3.32), \(H_{ik}(j\omega_1)\) is known since, it is the FRF-peak value obtained from the graphic. Additionally, the fractions with the corresponding complex conjugate residues and poles have been vanished since the relative damping coefficients, \(\zeta_r\) are quite small, meaning that the values for \(\sigma_n\) are also small, combined with the fact that the imaginary part of the denominator is increasing from the second mode until the last one (in the case of more degrees of freedom), because of the summation of frequencies, \(j(\omega + \omega_r)\).

Then, by evaluating (3.32) at the second frequency the remaining equation will be obtained, in order to solve the two equations with the two unknown values, the residues \(R_1\) and \(R_2\).

In the case of a SDOF system (figure 3.3), where the corresponding FRF shows only one peak (figure 3.5), the corresponding mass, \(m\), stiffness, \(k\), and damping, \(c\), can be determined from the only associated residue and pole as follows:
\[
m = -\frac{1}{2B\omega}
\]

\[
c = \frac{\sigma}{B\omega}
\]

\[
k = -\frac{\sigma^2 + \omega^2}{2B\omega}
\]

where the residue is expressed as \( R = Bj \) and the pole as \( s = \sigma + j\omega \).

### 3.4 Parametrically Excited Vibrations

In many cases, vibrations can mathematically be described by SDOF systems with a second-order linear differential equation with constant coefficients. Physically, this is reasonable since those coefficients correspond to properties such as mass, stiffness and damping, which in some cases remain unchanged with time.

However, there are cases where such coefficients can be dependent on time, bringing out an equation corresponding to what is known as vibrations induced by parametric excitations. As follows an explanation on Parametrically Excited Vibrations is given according to references [1] and [14].

In the case of Parametrically Excited Vibrations, the corresponding governing equation can be expressed as [14]:

\[
\ddot{x} + p_1(t)\dot{x} + p_2(t)x = F(t)
\]  

(3.36)

In Eq.(3.36) the presence of coefficients as functions depending on time can act as an excitation regardless the external force, \( F(t) \). The study of Eq.(3.36), when there is no external force and \( p_1(t) \) and \( p_2(t) \) are periodic functions with period \( T_0 \), is usually referred as Floquet Theory due to its author, who published it in 1883.
According to this theory, the two independent solutions, whose linear combination conduct to all possible solution of the homogeneous equation, are given by:

\[ x_i = e^{\alpha_i t} \phi_i(t) \quad i = 1, 2 \]  

(3.37)

where \( \phi_i(t) \) are periodic functions with period \( T_0 \), and the constant \( \alpha_i \) can be complex.

From Eq.(3.37) is noticed that if the real part of \( \alpha_i \) is positive, the solution is unbounded, consequently, leading to an unstable behavior of the system. On the other hand, if the imaginary part of this constant is not significantly small, the result is the product of a function with period \( T_0 \) with another function whose period can be different to this. Therefore, the solution can be an oscillation, damped or induced, whose period can be different to \( T_0 \).

An example of parametrically excited vibrations [1] is found in the oscillations of a pendulum when its anchorage point is not stationary and it is moving in the vertical direction (Figure 3.9). In fact, the vertical displacement of that point is who induces the vibration or oscillation in the pendulum conducting in determined cases to an unstable behavior characterized by large amplitudes which are significantly increasing in a short time.

![Figure 3.9 Oscillation of a pendulum with parametric excitations.](image)
Focus on the dynamic expression which governs the oscillation of the pendulum, we can obtain by considering the attachment point $A$ coincident with the origin of the fixed-coordinate system, the following:

$$J_A \ddot{\theta} + m a_A l \sin \theta + mg l \sin \theta = 0$$  \hspace{1cm} (3.38)

where:

- $J_A$, is the mass moment of inertia of the pendulum with respect to point $A$,
- $m$: is the pendulum’s mass,
- $a_A$: is the acceleration of the point $A$ (non-stationary attachment point),
- $l$: the length of the pendulum chord,
- $\theta$: the swept angle described by the pendulum.

For small oscillations, $\sin \theta$ can be approximated to $\theta$. In addition, assuming that the attachment point $A$ has a vertical harmonic displacement given by:

$$y_A = y_o \cos \omega t$$  \hspace{1cm} (3.39)

then, the acceleration in point $A$ is expressed as:

$$a_A = -\omega^2 y_o \cos \omega t$$  \hspace{1cm} (3.40)

where $\omega$, is the excitation frequency.

Since the mass moment of inertia for the pendulum is given by:

$$J_A = ml^2$$  \hspace{1cm} (3.41)

consequently, the equation (3.38) can be rewritten as:

$$\ddot{\theta} + \frac{g}{l} \left( 1 - \frac{\omega^2}{g} \right) y_o \cos \omega t \theta = 0$$  \hspace{1cm} (3.42)

As seen in Eq.(3.42), the coefficient multiplying with the swept angle, $\theta$, is depending on time, then describing an equation for parametrically induced vibrations.
The Eq.(3.42) takes the form of what is known as the Mathieu Equation as follows:

\[ y'' + (a - 2q \cos(2\tau))y = 0 \]  \hspace{1cm} (3.43)

where:

\[ \omega t = 2\tau \]
\[ a = \frac{4g}{l\omega^2} \]
\[ q = \frac{2y_o}{g} \]  \hspace{1cm} (3.44)

The advantage of expressing the system by using the Mathieu Equation remains in the fact that it is possible to see the range where its behavior is unstable. This is appreciated on the Mathieu Equation stability chart (Fig 3.10), whose shaded areas represent the unstable regions, depending on the values of \( a \) and \( q \), which are related to the frequency and the amplitude, respectively, of the movement which induced the vibrations (the attachment point for the example of the pendulum).

Figure 3.10 Mathieu Equation Stability Chart.
4. Modeling a Stay Cable

In cable-stayed bridges, parametric excitations are mainly associated to periodical displacements in the cable anchorages, which are located at the deck and the tower of the bridge. Such displacements are induced by traffic loads or wind loads acting on the bridge structure as shown in Figure 4.1. Then, tension oscillations changing on time are registered at the stay cables, which finally act as an external excitation producing vibrations. Mainly the vibrations are observed in-plane of the cable, but they can be also expected out-plane of the cable.

The mathematical background of parametrically excited vibrations on stay cables remains in the Mathieu Equation (3.43), where one of the coefficients in the corresponding governing differential equation, is changing on time, acting as an excitation, regardless the external force on the system. When deducing the governing equation for a stay cable, such parameter changing on time, is the tension.

Figure 4.1 Representation of a Cable-Stayed Bridge subjected to Parametric Excitations.
In order to study the vibrations on stay cables when they are induced by parametric excitations, different models have been introduced. In this thesis report those models presented by Pinto and Lilien [2], Sun et al. [3] and Gatulli, Paolone and D’Amore [4] are described below.

4.1 A stay cable with one anchorage free to move

In 1994, Pinto and Lilien [2] introduced a model presenting the stay cable as a continuous system with one of its anchorages free to move periodically, in order to reproduce the support motions, as shown in Figure 4.2.

They obtained, analytically, a non-linear Mathieu differential equation for the first generalized coordinate $Y$, considering the assumption that the excitation frequency was two times higher than the first natural resonance frequency of the cable (ratio=2), given by:

$$\ddot{Y} + b\dot{Y} + \omega^2_1 \left[1 + 2\delta \cos(2vt) + \frac{4}{3} \left(\frac{Y}{K}\right)^2 \right] Y = 0 \quad (4.1)$$
where $b$ is a damping coefficient, $\omega_1$ is the first resonance frequency of the cable, $\delta = \frac{X_d}{2X_0}$ and $K = (4/\pi)\sqrt{X_0l/3}$ . ($X_0$ and $X_d$ represent the initial stretch and the driving amplitude respectively; while $l$, is the cable length).

In order to guarantee the tautness or flatness of the cable, they used a condition for a non-dimensional parameter $\chi$, which must be much smaller than $2\pi ((\chi << 2\pi)$. This parameter pretends to involve the geometric and elastic properties of the cable as a function of the static tension $T_0$, the cable length $l$, the inclination angle $\theta$, the elasticity modulus of the cable $E$, the cable-cross section $A$, the cable mass per unit of length $m$, the acceleration of gravity $g$, as follows:

$$\chi^2 = \frac{EA(mgl \cos \theta)^2}{T_0^3}$$  \hspace{1cm} (4.2)

For example, cables of electrical overhead are associated with large values for $\chi^2$; while the opposite happens for stay cables.

After considering some simplifications, the analytical solution which described the cable motion was given by:

$$Y(t) = A\sin(\omega_1 t)$$ \hspace{1cm} (4.3)

where the amplitude $A$ was expressed as:

$$A = \frac{4}{\pi} \sqrt{\frac{X_0l}{3}} \sqrt{\left(\frac{X_d}{2X_0}\right)^2 - \left(\frac{b}{\omega_1}\right)^2}$$  \hspace{1cm} (4.4)

Furthermore, they established that during the transient response the amplitude of the vibration could reach the maximum value, given by the following:

$$A_{\text{max}} = \frac{2}{\pi} \sqrt{2lX_d}$$ \hspace{1cm} (4.5)
Although their analytical solution, their conclusive results were based on numerical simulations by using a software working with finite element method (FEM). They considered some practical or real cases for their simulations, finding larger amplitudes when the excitation was moving at the same or twice the first natural frequency of the cable, that means for the cases in which the frequency ratios were 1 or 2 (anchorage frequency/cable frequency), respectively.

However, in their simulations for larger stay cables like 440 m-long, leaded to significant conservative results given by amplitudes of 15m peak to peak. The authors introduced as a possible reason for presenting such large amplitudes the non-consideration of aerodynamic damping into their simulations.

It is also important to point out that the authors indicate the possibility of having a broad range of frequencies in which the anchorage can move inducing parametric excitation on cables, and not only when such frequency is the same or twice the first natural resonance of the cable. Depending on how large the excitation amplitude can be, such frequency range can cover all frequencies from about zero up to about twice the first resonance of the cable. Then, in the way that larger amplitude of excitation exists, parametric excitation can occur in some cases, regardless the frequency of excitation.

4.2 A stay cable interacting with the tower/deck of the bridge

In 2000, Sun et al. [3] developed an integrated model considering the behavior of the cable and tower/deck as depending on each other. The authors did not consider the incorporation of damping for the cable neither for the anchorage movement which will be associated to the pylon or deck of the bridge. In figure 4.3, is shown an adaptation of their model, where the damping from the pylon or deck is already considered.

As seen in figure 4.3, the influence of the pylon or deck of the bridge over the anchorage movement has been modeled by a mass $m$, a spring $k$, and a damper $c$, as considered in a SDOF system.
That means an anchorage motion depending on the displacement of the pylon or deck at the anchorage location. Based on the mechanical properties of the pylon/deck, a lumped mass system can be obtained, being in this case a SDOF system. In that way, only one resonance of vibration for the pylon/deck is considered, allowing establish the corresponding ratio between the resonances of cable and pylon/deck.

In real cases, cables, pylons and decks can vibrate at different resonance frequencies, however, based on the principle of superposition and modal analysis, each case can be treated separately and then the total displacement could be approximated by the summation of each of them.

Originally, from their model, they obtained a coupled system of two nonlinear differential equations, which describe the displacement amplitude for the cable and for the mass \( m \), which represents the anchorage motion due to the pylon or deck. The governing equation for the cable was based on the corresponding to a taut cable given by Eq.(3.2). Only the first mode shape of the cable was taken into account (See figure 3.2).

The consideration of parametric excitation was introduced by expressing the cable tension, according to the respective axial stress formulation [15], but as a function changing on time as follows:
where $A$ is the cable-cross section, $E$ the elasticity modulus of the cable, $l$ the cable length, $z_0$ the initial cable stretching, $X_2$ the displacement of the cable anchorage depending on time, and $\delta$ the extension of the cable induced by elastic deformation, changing also on time.

In other words, Eq.(4.6) represent the tension as a multiplication of the axial cable stiffness by its total displacement, as establishes the Hooke’s Law [15]. The extension, $\delta$, is interpreted as that one elastically induced when the transverse vibration of the cable, in $X_1$ direction, has occurred.

The estimation of $\delta$ is based on the assumption that taut cables show a ratio curvature/length very small ($\partial x_1 / \partial z << 1$). Then, by using trigonometric relationships, $\delta$, can be calculated by the following expression:

$$\delta = \int dx_1 = \int ds - dz$$

where $dx_1$ is the infinitesimal increment for the transverse displacement, $ds$, represents an infinitesimal arc of the cable; while $dz$ an infinitesimal part of the cable projected onto $z$-axis or reference axis for the cable vibration, as shown in figure 4.3 and figure 4.4.

By trigonometry, the magnitude of $ds$ can be written as:

$$ds = \sqrt{\left(\frac{\partial x_1}{\partial z} dz\right)^2 + dz^2} = \sqrt{1 + \left(\frac{\partial x_1}{\partial z}\right)^2} dz$$

Eq.(4.8) can be simplified by expressing $ds$ as a Mac Laurin series (Taylor expansion evaluated in zero) [8] as follows:

$$ds = 1 + \frac{1}{2}\left(\frac{\partial x_1}{\partial z}\right)^2$$
Then, replacing (4.9) into (4.7) gives us the following approximation for $\delta$:

$$\delta \approx \frac{L}{2} \left( \frac{\partial x_1}{\partial z} \right)^2 dz$$

(4.10)

Since, only the first mode shape for the cable is taken into account for the model, and considering that the origin of coordinate $x_1$ is coincident with the cable-middle point, the expression for the transverse cable vibration, based on Eqs.(3.2), (3.3) and (3.12), is rewritten as:

$$x_1(z,t) = X_1(t) \cos \left( \frac{\pi z}{l} \right)$$

(4.11)

where, $X_1(t)$ indicates the respective generalized coordinate, while the mode shape is expressed by a cosine in instead of a sine, because of the translation of the origin of coordinates to the cable-middle point.

Therefore, from the expression (4.11) the partial derivative of $x_1$ with respect to $z$ can be estimated and replaced into (4.10), bringing out the value for $\delta$ as follows:

$$\delta \approx \frac{\pi^2 X_1^2}{4l}$$

(4.12)

The tension, which can be expressed as a summation of the static, $T_0$ and dynamic component $T_d$ is, then, given by:
\[
T = T_0 + T_d = \frac{AE}{l} z_0 + \frac{AE}{l} \left[ X_2 + \frac{\pi^2 X_1^2}{4l} \right]
\] (4.13)

By considering the solution (4.11) and evaluating it into the governing equation for a taut cable given by Eq. (3.2) is obtained the following:

\[
\ddot{X}_1 + \left( \frac{\pi^2 T}{\rho A l^2} \right) X_1 = 0
\] (4.14)

where \( \rho A \), is the cable mass per unit of length.

Then by substituting the expression for the tension given by (4.13) into (4.14) we obtain:

\[
\ddot{X}_1 + \omega_1^2 \left[ 1 + \frac{X_2}{z_0} + \frac{\pi^2 X_1^2}{4l z_0} \right] X_1 = 0
\] (4.15)

where \( \omega_1 \) is the first natural frequency of the cable given by:

\[
\omega_1 = \frac{\pi}{l} \sqrt{\frac{T}{\rho A}}
\] (4.16)

The Eq.(4.15) represents the differential equation for the vibration amplitude of the cable; while for describing the motion of the mass \( m \), we have the following differential equation after applying the Newton Second’s Law:

\[
T_d + m \ddot{X}_2 + kX_2 = 0
\] (4.17)

Finally, by combining Eq.(4.15) and Eq.(4.17) is found the governing system of differential equations for the cable-pylon/deck system presented by Sun et al. [3], as follows:
\[ X_1 + \omega_1^2 \left(1 + \frac{X_2}{z_0} + KX_1^2 \right)X_1 = 0 \] 

\[ \ddot{X}_2 + \omega_2^2 X_2 + \frac{AE}{ml} Kz_0 X_1^2 = 0 \]

(4.18)

where \( K = \pi^2 / (4lz_0) \) and \( \omega_2 \) represents the resonance frequency for the mass \( m \) given by:

\[ \omega_2^2 = \frac{AE}{ml} + \frac{k}{m} \] 

(4.19)

As seen in the system of equations (4.18), any damping was not considered by the authors. They solved such system of differential equation by numerical approximations, considering only the case where the pylon/deck resonance frequency was two times higher than the cable resonance frequency (ratio=2), since this has been reported as the most critical condition in parametric excitation on stay cables. As a consequence of their model, an undamped solution for cable and pylon/deck was obtained, in which the cable amplitude reached 1m when the mass \( m \), associated to the pylon or deck was excited with an amplitude of 10 cm.

However, as following, a system of equations is presented were the inherent damping of the cable and the damping introduced for the pylon or deck are taken into account, as well as an external force acting on the mass \( m \), simulating any external excitation force acting onto the pylon or bridge deck:

\[ \ddot{X}_1 + 2\zeta_c \omega_1 \dot{X}_1 + \omega_1^2 \left(1 + \frac{X_2}{z_0} + KX_1^2 \right)X_1 = 0 \] 

\[ \ddot{X}_2 + 2\zeta_p \omega_2 \dot{X}_2 + \omega_2^2 X_2 + \frac{AE}{mL} Kz_0 X_1^2 = \frac{F_2}{m} \] 

(4.20)
where $\zeta_c$ and $\zeta_p$ represent the relative damping for the cable and for the pylon or deck, respectively.

The system of equations 4.18 and 4.20 are ordinary differential, meaning that they are only depending on one variable [8], the time. Then, they can be solved by applying any of different numerical methods, like Runge-Kutta or Milne [16].

### 4.3 A stay cable supported by a cantilever beam

The last model presented in this thesis report corresponds to that one introduced by Gatulli, Paolone and D’Amore [4], in 2002. Such model was based only in the interaction between cable and the bridge deck, which was modeled as a cantilever beam. Figure 4.5 shows the model presented by them.

Mainly, this model has been chosen as other reference about modeling a stay cable and as a consequence, details like the formulation of the governing equations are not considered in the present report, since its natural complexity when considering and expressing those equations depending on time and space coordinates, at the same time.

![Figure 4.5 A Stay Cable supported by a cantilever beam.](image)
From this model, the authors reported a non-linear interaction between the cable and deck described by a quadratic coupling. They also indicated that such non-linear phenomena was increased when the ratio between the deck and cable resonances were about 0.5 and 2.
5. Methodology

As was explained previously, the aim of this thesis is to study by experimental and numerical analysis the vibration on stay cables subjected to parametric excitations, taking as a reference the Øresund Bridge. Therefore, a scaled model of this structure was employed in order to carry out the corresponding experimental tests, as well as numerical simulations based on the model which considers the interaction between pylon/deck and cable explained in chapter 4.2. The numerical solutions were obtained by using subroutines in the engineering software Matlab, based on the description of any mechanical system by its transfer function, and considering the non-linear parameters as external forces.

5.1 The Øresund Bridge

The Øresund Bridge is a structure approximately 8 km long which together with an artificial island which covers a length of 4 km, and a submarine tunnel 4 km long, gives that is known as the Øresund Link. This 16 km-link connects the road and the rail system between the cities of Malmö at the south of Sweden and Copenhagen, Denmark, through the Øresund channel.

![Figure 5.1 The Øresund Bridge (High Bridge) connecting Sweden and Denmark (Obtained from www.bridgephoto.dk)](image)
In the Øresund Bridge, there is a high bridge at the middle and two approach bridges, each of them located at each side of the high bridge. The high bridge (See figure 5.1) is composed by two pairs of towers or pylons 204 meters high above the sea level, supporting the deck by a harp-shape arrangement of stay cables. In total, there are 80 pairs of stay cables, where the longest is 262 m against 52 m-long for the shortest one.

The resulting free span for navigation is about 490 m long with 55 m high of clearance (measured from the sea level to the deck-lower level). Such span distance is, by far, the longest in the world for a cable-stayed bridge which allows road and rail traffic [17].

5.2 A Scaled Model of the Bridge

For developing experimental analysis, the respective tests were carried out on a scaled model (1:200) of the high bridge of the Øresund Bridge. Such model was made of Aluminum, and as seen in its representation in figure 5.2, the model was simplified by considering one pair of pylons. Since the largest amplitudes in the real bridge have been observed on the longest stay cables, the model took into account only those as shown in figure 5.2.

![Figure 5.2 Representation of the scaled model used for experimental testing.](image)

Figure 5.2 Representation of the scaled model used for experimental testing.
The cables were replaced by piano strings made of steel where one of its ends was fixed to the pylon while the remaining end was attached to a screw (See detail in figure 5.2) in order to adjust and obtain the required initial tension by tightening or loosing such screw.

5.3 Experimental Setup

The experimental setup was based on studying the scaled model of the Øresund Bridge subjected to an excitation force at one of the pylon, since this is structurally more flexible than the bridge deck. In order to observe the induced vibration, only one string was considered, specifically that one attached to the pylon where the external force was introduced, as shown in figure 5.3.

![Figure 5.3 Scaled Model of the bridge excited at one of the pylon by a shaker.](image)

As expected, several devices and instruments were employed in order to develop the corresponding tests. They are mentioned below; while some of them can be appreciated in figure 5.4:

- **A Shaker:** with a capability to transmit a force of up to 50 N, it was the device used to excite the structure at one of its pylon.
- A Force Transducer: to measure the force transmitted by the shaker.
- A Signal Generator: to define electronically the force to be introduced.
- A Power Amplifier: to magnify the magnitude of the force, originally defined by the signal generator and finally transmitted by the shaker.
- An Accelerometer: to measure the response and estimate from the measured acceleration, the displacement at the top of the pylon, which means the displacement of the movable end of the string.
- A Laser Vibrometer: to estimate the displacements of the string, from its measured velocity.
- A Data Acquisition Card: to collect the data, which were introduced and measured onto the system.
- A Computer: to store and to interpret the data by using the engineering software Matlab.

Figure 5.4 Part of devices and instruments used for the experimental setup.

It is important to point out that the measurement of vibration on strings, which are characterized by presenting small diameter, means a challenge, since the use or attachment of any instrument on them like an accelerometer is not possible. In the present case, the alternative was applying a non-contact measurement method given by the use of a laser vibrometer.
5.4 Characterization of the Pylon and String

Since the numerical model studied in this thesis was based on the interaction between string and pylon or deck according to Eq.(4.20), the displacement of the movable end of the string required to be characterized. From the scaled model, such movable end was found at the top of the pylon because of its higher flexibility to move when it is compared to the bridge deck.

Therefore, a proper characterization of the motion at the top of the pylon was required in order to describe its vibration by obtaining a SDOF system as implies the numerical model selected and represented by figure 4.3. Obviously, the natural resonance of the pylon must be determined, since the experiment must accomplish the condition of having a resonance frequency for the pylon at one half of the resonance frequency of the string, as in the real case for the longest stays at the Øresund Bridge.

For the present scaled model, the first-pylon resonance frequency was found about 50 Hz. Then, the first natural resonance for the string could be adjusted as double of the pylon, according to its mathematical expression Eq.(3.14), by choosing its diameter, which finally was 0.2 mm, and by adjusting its initial tension, since its length was already defined as 1.33 m, according to the scaling factor.

The characterization consisted in obtaining experimentally the frequency response function (FRF) corresponding to the top of the pylon. This practice is called experimental modal testing and in this case was determined by the shaker excitation method [11]. This method implies the application of a random force (varying arbitrary on time) which is defined over a frequency range. Then, the ratio between the response measured by the accelerometer and this force, together expressed in the frequency domain, will show the natural frequencies of the system where the FRF presents its peak values, as well as the possibility to extract the corresponding modal parameters as was explained in chapter 3.3.

This experimental practice in general requires the excitation and the measurement of the corresponding response over convenient points on any mechanical system. That means, where the response will present its maximum amplitude, after exciting. In the present case, the best point for
exciting the structure would be at the top of the pylon, however due to limitations in reaching that height for positioning the shaker, the excitation point was located at 25 cm from the top (See figure 5.5). Such height was good enough to obtain a good response for the pylon.

5.5 Experimental Modal Testing by using Shaker Excitation.

By considering only one natural frequency like the first resonance of the pylon, the mass $m$, the damping $c$ and the stiffness $k$ could be calculated from Eqs. (3.33 to 3.35). Then, the displacement at the top of the pylon and, as a consequence, the displacement of the movable cable end is equivalently described by a SDOF system as required by the numerical model in study.

In figure 5.5 is shown the pylon subjected to the force transmitted by the shaker, as well as the corresponding accelerometer measuring the response at the top of the pylon.

Since the characterization test was carried out under the required frequency ratio between pylon and string (ratio=0.5), the same methodology used to
characterize the pylon was simultaneously applied for the string in order to estimate from the corresponding FRF, mainly the string damping, since this is the only parameter required for the numerical model.

5.5 Response of the system under sinusoidal excitation.

In order to study the vibrations induced by parametric excitations on the string, it was applied a sinusoidal force onto the pylon. Then, a maximum response from the pylon could be reached by applying a force whose frequency matches one of their natural frequencies. In the present case, the frequency force matched the first-pylon resonance frequency (about 50 Hz).

A maximum response from the pylon is desirable since, larger amplitudes for the displacements at the movable end of the cable are strongly associated to the phenomena of parametric excitation, even regardless of the excitation frequency [2].

Once the force frequency was chosen to be coincident with the first resonance frequency of the pylon, the amplitude of the force was gradually increased from a lower to a higher magnitude. This practice basically looks for detecting the starting point or force in which the parametric excitation phenomena can be significantly observed by noticing a non-linear behavior in the string vibration. This behavior is clearly characterized by a sudden and significant increment or jump in its amplitude.
6. Results

The obtained results are presented according to the methodology explained in the previous chapter.

6.1 Characterization of the Pylon and String

The shaker excitation method was employed in this step in order to obtain the FRF for the pylon and string under three different amplitudes of random force. This procedure represents an useful tool for recognizing non-linear system, in which usually the amplitude of the response and/or the resonance frequencies will be dependent on the excitation amplitude [18].

The three different levels of force amplitude were classified according to its root mean square (RMS) [11], and they were corresponding to the following values: about 3N, 6N and 13 N. The results are presented below.

![Figure 6.1 Different FRF’s at the top of the pylon depending on force amplitude.](image)

In figure 6.1 the FRF’s are expressed as a ratio between acceleration and force. This relation is called accelerance. In this figure, the FRF’s obtained
experimentally are represented by the marks indicated at the legend. For each level of amplitude force, two or three tests were carried out. On the other hand, by extracting the modal parameters from the experimental FRF’s, one estimated curve fitting each level of force was obtained and represented by a line. In table 6.1 are presented the corresponding extracted modal parameters for each level of force.

Table 6.1 Modal Parameters for the pylon depending on the level of force

<table>
<thead>
<tr>
<th>RMS - Force [N]</th>
<th>m [kg]</th>
<th>c [N.s²/m]</th>
<th>k[N/m]</th>
<th>ζ [%]</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0.9167</td>
<td>0.7033</td>
<td>92809</td>
<td>0.12</td>
</tr>
<tr>
<td>6</td>
<td>0.9045</td>
<td>0.7951</td>
<td>91484</td>
<td>0.14</td>
</tr>
<tr>
<td>13</td>
<td>0.9441</td>
<td>1.0007</td>
<td>95273</td>
<td>0.17</td>
</tr>
</tbody>
</table>

The experimental data regarding the string was collected simultaneously when exciting the pylon at the different levels of force mentioned above. In figure 6.2 are observed such FRF’s, expressed as a ratio between velocity and force, which is called mobility. From the experimental FRF’s the string relative damping, ζ_s, was estimated as 0.06% varying slightly around this value for the different levels of force applied.

![Mobility FRF for String](image)

**Figure 6.2 FRF’s for the string depending on the amplitude force.**
Furthermore, in figure 6.3 is represented the first mode shape of vibration for the pylon, which occurs at around 50.6 Hz.

![Figure 6.3 First mode shape of vibration for the pylon at 50.6 Hz](image)

### 6.2 Response of the system under sinusoidal excitation

Once the characterization of the pylon and string were done, the evaluation in Matlab by using numerical subroutines into the corresponding numerical model of the system was possible. In the table 6.2 are gathered all parameters and values required for such model, according to Eq. (4.20).

**Table 6.2 Parameters for the numerical model.**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>String density, $\rho$ [kg/m$^3$]</td>
<td>7800</td>
</tr>
<tr>
<td>String Diameter [m]</td>
<td>$2\times10^{-4}$</td>
</tr>
<tr>
<td>String Cross Section, $A$ [m$^2$]</td>
<td>$3.1416\times10^{-8}$</td>
</tr>
<tr>
<td>String Elasticity Modulus, $E$ [N/m$^2$]</td>
<td>$205\times10^9$</td>
</tr>
<tr>
<td>String Length, $l$ [m]</td>
<td>1.33</td>
</tr>
<tr>
<td>String Relative Damping, $\zeta_s$ [%]</td>
<td>0.06</td>
</tr>
<tr>
<td>Pylon Equivalent Mass, $m$ [kg]</td>
<td>0.92</td>
</tr>
<tr>
<td>Pylon Equivalent Stiffness, $k$ [N/m$^2$]</td>
<td>92809</td>
</tr>
<tr>
<td>Pylon Relative Damping, $\zeta_p$ [%]</td>
<td>0.12</td>
</tr>
</tbody>
</table>
In figure 6.4 is shown the sinusoidal force applied onto the pylon. Its frequency was 50.6 Hz being coincident with the first resonance frequency of the pylon.

In order to compare the experimental results with the simulations from Matlab, the results will be shown simultaneously as follows:

Figure 6.4 External Force applied at 50.6 Hz.

Figure 6.5 Experimental and numerical displacement for the Pylon
Figure 6.6 Experimental and numerical displacement for the String

Figure 6.7 Experimental and Numerical Spectrum for the Pylon.
Figure 6.8 Experimental and Numerical Spectrum for the String.

Figure 6.9 Experimental and Numerical Displacement-Force relationship for the pylon.
Figure 6.10 Experimental and Numerical Displacement-Force relationship for the string.
7. Discussion and comparison of results

7.1 Characterization of Pylon and String

From the characterization of the pylon and the string, it could be noticed peak values for the corresponding FRF’s depending on the force amplitude introduced onto the system, more clearly observed for the pylon. That property is expected for non-linear system [18], as it is the case in study. In fact, from table 6.1 is observed how the inherent damping of the pylon is increasing when the force amplitude is also increased.

It was also observed how the mode shape of vibration of the pylon is influenced by its connection to the bridge deck, bringing sort of inflexion point around this height and reducing the freedom to vibrate for the pylon from the deck connection until its clamped end.

The first resonance for the string was adjusted to 100.3 Hz against 50.6 Hz for the first resonance frequency of the pylon, giving a ratio \( \frac{\omega_{\text{pylon}}}{\omega_{\text{string}}} \) equal to 0.505, which is close enough to the condition expected to be evaluated.

7.2 Response of the system under sinusoidal excitation

From the experiment developed here, it can be established that the parametric excitation phenomena could be appreciated under the studied condition \( \frac{\omega_{\text{pylon}}}{\omega_{\text{string}}} \approx 0.5 \). This phenomena was clearly identified by presenting a sudden and significant increment in the string vibration amplitude, when increasing gradually the amplitude force (See figure 6.6). Such instability was observed when the amplitude for the sinusoidal force reached a value around 1.6 N, which meant a corresponding displacement amplitude for the pylon at its top location about 1.2 mm. (See figures 6.9 and 6.10).

Additionally, the main string vibrations were found at its first natural frequency (100.3 Hz), regardless of the excitation frequency, which was
different (about 50 Hz). This fact agrees with the statement basically assumed when parametric excitation on cable occurs [2].

From the numerical results, it was observed higher predicted amplitudes for the displacement of pylon and string, specially, during the instable process where the string reaches its maximum amplitude. In the case of the string, the comparison between numerical and experimental values for the amplitudes is not applicable since the configuration design of the Laser Vibrometer allows measure vibrations in the parallel direction of the laser beam, which means out-plane vibrations for the string. Then, the measurement by using the laser worked as a reference to estimate the resonance of vibration and its pattern, as the non-linearity observed here on the string.

In spite of the limitation to compare directly the numerical and experimental amplitudes for the string, such large amplitudes about 5 cm predicted by the numerical analysis were not reached, by far, according to the visual appreciation.

One reason which could explain such results is the fact that the numerical model does not involve directly the mechanical properties of the cable, based on its total energy to be deformed. If we take back the expression (4.6) which describes the tension changing on time and depending on: the initial stretch, the excitation amplitude and the elastic extension induced by the lateral vibration on the cable as:

\[
T = \frac{AE}{l} \left[ z_0 + X_2 + \delta \right]
\]

we noticed that after pre-stressing the cable, there will be a remaining amount of deformation energy which will be consumed by the other two displacements under which is subjected the cable. Depending on how large is the pre-tension and how flexible the cable is, the transverse vibration of the cable will be defined by the remaining total elastic energy after its anchorage is moving, if we do not consider plastic deformation and neither the following process until the fracture is reached.

In other words, the cable presents itself a condition which limits its freedom to vibrate and be elongated by the anchorage displacement, after it has been pre-stressed. As a consequence, it would be convenient to measure the
mechanical properties of the cable under static and dynamic conditions. In that way, properties such as yielding and ultimate strength, the total percentage of elastic and plastic energy of deformation, as well as the strength to fatigue and how its flexibility is reduced along its dynamic loading, could be taken into account to define finally the freedom for the cable to vibrate and as a consequence to estimate amplitudes closer to the reality.

The explanation given above could be also considered for the numerical results obtained by Pinto and Lilien [2], when studied a stay cable of 440 m long and obtained an amplitude peak to peak of 15 m. Possibly, the considerations of the mechanical properties of the cable as indicated above, could fix such predicted amplitude.

Although the differences between the amplitudes obtained numerically and experimentally, the numerical model applied in this thesis work reveals properly the starting point in which parametric excitation is significantly appreciated by a sudden jump in the string amplitude. That meant, the simulations could conduct to predict properly that instable condition given by the minimum required excitation amplitude at the top of the pylon, in order to obtain the largest vibration amplitude for the string. This starting point was reached when the force amplitude was around 1.6 N, as was shown in figure 6.10.

7.3 The effect of the ratio pylon resonance-cable resonance.

In order to compare experimentally the effect of the ratio between the resonance frequencies of string and pylon, two cases are taken into account: when the ratio $\omega_{pylon} / \omega_{string}$ is 2, case which was already treated by Abu-Iseifan and Paulson [6] on the same scaled model of the bridge, and when such ratio is 0.5, corresponding to the case studied in this thesis. The comparison is based on the minimum required excitation amplitude force at the top of the pylon in order to create the instability on the string characterized by its sudden increment in its displacement. Additionally, it has been considered the non-dimensional parameter $\chi$, used by Pinto and Lilien [2] which in some way guarantee the flatness or tautness of the cable, when such parameter is less than $2 \pi$. 
Table 7.1 Starting Conditions for Parametrically induced Vibration on Strings

<table>
<thead>
<tr>
<th>$\omega_{pylon}/\omega_{string}$</th>
<th>Force Amplitude [N]</th>
<th>$\chi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.35</td>
<td>0.1</td>
</tr>
<tr>
<td>0.5</td>
<td>1.6</td>
<td>0.003</td>
</tr>
</tbody>
</table>

According to table 7.1, in both cases the flatness or tautness of the string was guaranteed since the parameter $\chi$ was much lower than $2 \pi$.

The criteria usually applied to determine which condition is more critical according to other research works [2] and [5], was based on how large the amplitude was obtained for the cable. In such cases, that was found when the ratio was 2 in instead of having 1 or 0.5, giving for the latter case, the smaller amplitude.

Equivalently, the table 7.1 confirms the fact that establishes a more critical condition when the ratio $\omega_{pylon}/\omega_{string}$ is 2, since it was required a 4.5 times much smaller force amplitude at the top of the pylon to generate the instable vibration of the string than in the case studied here, when such ratio is 0.5. Then, for the condition $\omega_{pylon}/\omega_{string} = 2$, the cable will be more prone to vibrate under parametric excitation.
8. Conclusions and Recommendations

After carrying out the corresponding numerical and experimental analysis, the present thesis work arrives to the following conclusions:

1. Vibrations on the string induced by parametric excitations were appreciated under the condition $\frac{\omega_{pylon}}{\omega_{string}} = 0.5$. Such vibrations were clearly identified by the instable displacements of the string, mainly characterized by a sudden increment in its amplitude for a specific value of the introduced force.

2. Higher amplitudes than those estimated experimentally were predicted for both string and pylon according to the numerical analysis carried out in the present work. That was also reported for other researchers whose simulations estimated much larger amplitudes than they could be expected, when considering long stay cables.

3. Although these differences in amplitudes obtained by experiments and simulations, the numerical analysis used in the present work predicted properly the starting point in which the instable behavior for the string is reached.

4. It was experimentally determined that a string is more prone to vibrate under parametric excitations when the resonances ratio $\frac{\omega_{pylon}}{\omega_{string}}$ is 2, than when this ratio is 0.5, since bigger excitation amplitudes are required in the latter case in order to reach the instable behavior and large amplitudes for the string.

Furthermore, some recommendations are also given in order to be considered for future researches which could take the present thesis work as a reference, as follows:

1. Considering the conservative results from numerical analysis based on the model employed here, as well as in other research works mentioned in the report, it is recommended to evaluate experimentally the mechanical properties of the string or cable and introduce those into them, in order to probably obtain amplitudes closer to the reality.
2. Consider the studied condition on cables by evaluating the other two numerical models mentioned in the report.

3. Consider the study of several resonance ratios between excitation and cable, other than those punctual cases mostly indicated in the literature on Parametric Excitation on cables.

4. Evaluate experimentally and numerically the cable behavior subjected to passive methods to mitigate vibration, like for example: by using cross ties and tuned-mass dampers.
9. References


17. [www.oeresundbron.dk](http://www.oeresundbron.dk). *The Link:” The Bridge”.*
