Inexact programming

Muhammad Yasir Mahmood

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School of Engineering
Department of Mathematics and Sciences
Blekinge Institute of Technology, Sweden
Supervisor: Professor Elisabeth Rakus-Andersson
Examiner: Dr. Claes Jogreus
Contact Information:

Author:
Muhammad Yasir Mahmood
Email: yasir_paec@yahoo.com

Supervisor:
Professor Elisabeth Rakus-Andersson
Department of Mathematics and Sciences
School of Engineering, BTH
Blekinge Institute of Technology, Sweden
Email: elisabeth.andersson@bth.se

Examiner:
Dr. Claes Jogreus
Department of Mathematics and Sciences
School of Engineering, BTH
Blekinge Institute of Technology, Sweden
Email: claes.jogreus@bth.se
Abstract

Two types of fuzzy linear programming i.e. fuzzy number linear programming and interval number linear programming are used for optimization problems. In interval form of linear programming we convert the inequalities from the feasible region, containing intervals as coefficients, to two groups of inequalities characterized by real, exact coefficients values. Then classical programming has been used to achieve an optimal solution in the feasible region. In fuzzy number linear programming, α-cuts and LR forms of fuzzy numbers as coefficients have been used to find optimal solution in the feasible region. Finally the numerical examples and their solutions are attached to provide explanations of procedures.
Acknowledgements

All thanks to Almighty Allah, the creator and the lord of the universe, the most merciful and beneficent, who enabled me to complete this thesis successfully.

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Muhammad Yasir Mahmood
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Chapter 1

Introduction

In the current thesis we propose the discussion about some approaches to the inexact programming in the linear form [4, 5, 6, 8], which differs from forms of fuzzy linear programming [1, 10].

The inexact program model preserves the real and exact values of coefficients included in the objective function, whereas the coefficients of the feasible region have the inexact character. The linear maximization or minimization problems often refer to either evaluations of some profits or times consumed by technological or economical processes.

Sometimes we experience difficulties with the exact appreciations of coefficient values, which take place in the feasible region. We cannot exactly evaluate, e.g. the consumption of resources in several production processes as well as we are not able to formulate a precise number of units concerning the accessible amount of resources. These amounts depend on conditions of machines and conditions of stores.

We thus feel that a kind of inexact information, leaving a certain capacity space for the feasible region coefficients, should be introduced into the feasible region of a program to make it not so strict.

In the first trial to insert the inexact information into the feasible region we adopt its coefficients in the interval form. The region is split in two sub regions possessing real coefficients by using the mixture of inequalities “less than or equal” and “greater than or equal” due to the study made by Soyster [4, 5]. The program obtains the form of a classical linear program, which is solved by classical algorithms like the simplex method. We can treat this stage as the first one in the process of inserting vague information in the feasible region.

We also make another trial of improving the grade of imprecision in the feasible region by introducing its coefficients as fuzzy numbers, when using α-cuts forms of fuzzy numbers or L-R forms of fuzzy numbers. We convert the fuzzy inequalities of the feasible region to the interval inequalities of the linear program [4, 5, 6, 8, 10]. Nevertheless this program has been already found in the first stage of the study. We return to classical algorithms of solving the linear program obtained from the interval form.
Chapter 2

Interval-valued Inexact Programming

2.1 Simplex Method

Simplex method is used for solving the linear programming problem to get an optimal solution. It is an iterative method for solving a system of linear equations. The algorithm ends up when it either optimizes the objective or becomes infeasible. Simplex method is applied to linear equations of the standard form

\[
\text{Min } Z = CX
\]

subject to:

\[
AX \geq b, \ X \geq 0
\]

or

\[
\text{Max } Z = CX
\]

subject to:

\[
AX \leq b, \ X \geq 0
\]

i.e.

\[
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \leq b_1
\]

\[
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \leq b_2
\]

\[
\ldots
\]

\[
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \leq b_m
\]

where \(X = (x_1, x_2, ..., x_n)\) are variables, \(C = (c_1, c_2, ..., c_n)\) are the coefficients of the objective function, \(A\) is an \(m \times n\) matrix and \(b = (b_1, b_2, ..., b_m)\) are constants.

Explanation:

In max \((b_1x_1 + b_2x_2 + \ldots + b_nx_n)\), \(x_1, x_2, ..., x_n\) are unknown units of articles \(v_1, v_2, ..., v_n\), \(b_1, b_2, ..., b_n\) are costs for selling of one unit of \(x_1, x_2, ..., x_n\), and \(c_1, c_2, ..., c_m\) are accessible units of resources \(R_1, R_2, ..., R_m\).

The set of inequalities

\[
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \leq c_1
\]

\[
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \leq c_2
\]

\[
\ldots
\]

\[
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \leq c_m
\]
is called a feasible region.

**Example 2.1 (Problem of maximization)**

A carpenter makes three wood toys of types (A, B, C). The required time for cutting, polishing and packaging is given in Table 2.1.

Table 2.1 The consumption of time in three processes to make one unit of type A, B & C

<table>
<thead>
<tr>
<th>Processing</th>
<th>Type A (time to make one unit of A)</th>
<th>Type B (time to make one unit of B)</th>
<th>Type C (time to make one unit of C)</th>
<th>total available time (Minstes)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cutting</td>
<td>2</td>
<td>4</td>
<td>3</td>
<td>12000</td>
</tr>
<tr>
<td>Polishing</td>
<td>1/3</td>
<td>1/3</td>
<td>2/3</td>
<td>4600</td>
</tr>
<tr>
<td>Packaging</td>
<td>1/2</td>
<td>1/3</td>
<td>1/2</td>
<td>2400</td>
</tr>
<tr>
<td>Profit (SEK)</td>
<td>13</td>
<td>8</td>
<td>17</td>
<td></td>
</tr>
</tbody>
</table>

How many units of wood toys of every type should be made to get a maximum profit?

**Solution:**

Let $x$, $y$ and $z$ represent the number of units of types A, B and C respectively.

The objective function is given by

$$\text{Max } P = \text{Max } (13x + 8y + 17z)$$

subject to:

$$2x + 4y + 3z \leq 12000$$

$$(1/3)x + (1/3)y + (2/3)z \leq 4600$$

$$(1/2)x + (1/3)y + (1/2)z \leq 2400$$

where $x \geq 0$, $y \geq 0$, $z \geq 0$.

We solve the problem by simplex tabular method.

Let $s_1, s_2, s_3$ be the slack variables and $P$ be the objective value of the optimum solution.
Optimal solution: \( P = 72000 \) SEK; \( x = 2400, y = 0, z = 2400 \).

### 2.2 Dual Simplex Method

In dual simplex method we convert the primary problem into dual. In this way we revert either infeasible or unbounded primal problem (i.e. the original form of the optimization problem) into optimal basic solution. Dual simplex method starts from a dual feasible basic and works towards primal feasibility.

Practically if we have a problem of minimizing the cost (primary problem), we convert the primal into the maximizing the profit with the help of following algorithm.

\[
\text{Min} \left( c_1y_1 + c_2y_2 + \cdots + c_my_m \right)
\]

with restrictions

\[
a_{11}y_1 + a_{21}y_2 + \cdots + a_{m1}y_m \geq b_1
\]

\[
a_{12}y_1 + a_{22}y_2 + \cdots + a_{m2}y_m \geq b_2
\]

...  

\[
a_{1n}y_1 + a_{2n}y_2 + \cdots + a_{mn}y_m \geq b_n
\]
which is equivalent to

\[
\text{Max } (b_1x_1 + b_2x_2 + \cdots + b_nx_n)
\]

with restrictions

\[
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leq c_1
\]

\[
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \leq c_2
\]

\[
\quad \cdots
\]

\[
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \leq c_m.
\]

**Example 2.2 (Problem of minimization)**

A small juice producer has two juice units. One unit which is old is called Unit 1 and other unit 2 is newly purchased and modern. The operating cost of Unit 1 is 2000 SEK per hour. As the result of this operation, 40 liters juice of high quality, 30 liters juice of medium quality and 20 liters juice of low quality is obtained per hour. The operating cost of Unit 2 is 2500 SEK per hour, and it can produce 30 liters juice of high quality, 40 liters juice of medium quality and 50 liters juice of low quality per hour.

The juice company received an order of 2500 liters juice of high quality, 2700 liters juice of medium quality and 3000 liters juice of low quality. How many hours are required to fulfill the orders with minimized cost?

**Solution:**

Let \( x \) and \( y \) represent the number of hours for the operation of two units.

We state a program

\[
\text{Min } P = \text{Min } (2000x + 2500y) \quad \text{(Objective Function)}
\]

subject to:

\[
40x + 30y \geq 2500 \quad \text{(High quality)}
\]

\[
30x + 40y \geq 2700 \quad \text{(Medium quality)}
\]

\[
20x + 50y \geq 3000 \quad \text{(Low quality)}
\]

\[
x \geq 0 \text{ and } y \geq 0.
\]

We convert the program stated above into the maximization problem by utilizing the property of dualities between the min and max programs. We thus obtain
Max Q = Max (2500u + 2700v + 3000w)

subject to:

\[40u + 30v + 20w \leq 2000\]
\[30u + 40v + 50w \leq 2500\]
\[u \geq 0, \ v \geq 0, \ w \geq 0.\]

We solve the problem by simplex tabular method.

Let \( s_1, s_2 \) be the slack variables and Q be the objective value of the optimum solution.

Tableau # 1

<table>
<thead>
<tr>
<th>u</th>
<th>v</th>
<th>w</th>
<th>s_1</th>
<th>s_2</th>
<th>Q</th>
</tr>
</thead>
<tbody>
<tr>
<td>40</td>
<td>30</td>
<td>20</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>30</td>
<td>50</td>
<td>50</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>-2500</td>
<td>-2700</td>
<td>-3000</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

Tableau # 2

<table>
<thead>
<tr>
<th>u</th>
<th>v</th>
<th>w</th>
<th>s_1</th>
<th>s_2</th>
<th>Q</th>
</tr>
</thead>
<tbody>
<tr>
<td>28</td>
<td>14</td>
<td>0</td>
<td>1</td>
<td>-2/5</td>
<td>0</td>
</tr>
<tr>
<td>3/5</td>
<td>4/5</td>
<td>1</td>
<td>0</td>
<td>1/50</td>
<td>0</td>
</tr>
<tr>
<td>-700</td>
<td>-300</td>
<td>0</td>
<td>0</td>
<td>60</td>
<td>1</td>
</tr>
</tbody>
</table>

Tableau # 3

<table>
<thead>
<tr>
<th>u</th>
<th>v</th>
<th>w</th>
<th>s_1</th>
<th>s_2</th>
<th>Q</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1/2</td>
<td>0</td>
<td>1/28</td>
<td>-1/70</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1/2</td>
<td>1</td>
<td>-3/140</td>
<td>1/35</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>50</td>
<td>0</td>
<td>25</td>
<td>50</td>
<td>1</td>
</tr>
</tbody>
</table>

Optimal solution: \( Q = 175000 \) SEK; \( u = 250/7, \ v = 0, \ w = 200/7. \)

2.3. Inexact programming with intervals in the feasible region of LPP (linear programming problems)

2.3.1 Algorithm for maximization programs with interval coefficients in the feasible region

This algorithm is applied to the linear programming problem of the form

Max \( \sum_{j=1}^{n} c_j x_j = \text{Max} \ (c_1 x_1 + c_2 x_2 + \cdots + c_n x_n) \)

where \( c_j \in \mathbb{R}, \ j = 1, 2, \ldots, n, \) are the coefficients of the objective function.

We add the constraints

\[ \sum_{j=1}^{n} [\underline{a}_{ij}, \bar{a}_{ij}] x_j \subseteq [\underline{b}_i, \bar{b}_i], \]
where $X = (x_1, x_2, ..., x_n)$ are variables of the problem, $a_{ij}$ are the lower borders and $\bar{a}_{ij}$ are the upper borders of the intervals being the coefficients in the left side of the feasible region. The values $\underline{b}_i, \overline{b}_i$ constitute lower and upper borders for the intervals of the right side of the feasible region restrictions [4, 5, 8].

All upper and lower borders are real numbers. Now we consider the feasible region which has the intervals as the coefficients. We note it as a set of inequalities

$[a_{11}, \bar{a}_{11}] x_1 + [a_{12}, \bar{a}_{12}] x_2 +...+ [a_{1n}, \bar{a}_{1n}] x_n \leq [\underline{b}_1, \overline{b}_1]$

$[a_{21}, \bar{a}_{21}] x_1 + [a_{22}, \bar{a}_{22}] x_2 +...+ [a_{2n}, \bar{a}_{2n}] x_n \leq [\underline{b}_2, \overline{b}_2]$

$...$

$[a_{m1}, \bar{a}_{m1}] x_1 + [a_{m2}, \bar{a}_{m2}] x_2 +...+ [a_{mn}, \bar{a}_{mn}] x_n \leq [\underline{b}_m, \overline{b}_m]$

From above we create the program [4, 5, 8]

Max ($c_1 x_1 + c_2 x_2 +...+ c_n x_n$)

$a_{11} x_1 + a_{12} x_2 +...+ a_{1n} x_n \geq \underline{b}_1$

$a_{21} x_1 + a_{22} x_2 +...+ a_{2n} x_n \geq \underline{b}_2$

$...$

$a_{m1} x_1 + a_{m2} x_2 +...+ a_{mn} x_n \geq \underline{b}_m$

and

$\bar{a}_{11} x_1 + \bar{a}_{12} x_2 +...+ \bar{a}_{1n} x_n \leq \overline{b}_1$

$\bar{a}_{21} x_1 + \bar{a}_{22} x_2 +...+ \bar{a}_{2n} x_n \leq \overline{b}_2$

$...$

$\bar{a}_{m1} x_1 + \bar{a}_{m2} x_2 +...+ \bar{a}_{mn} x_n \leq \overline{b}_m$.

In that way the feasible region possessing coefficients in the form of intervals was converted to a set of inequalities in which the coefficients are real values. The linear program can be solved by the simplex tabular method.

**Example 2.3 (Problem of maximization)**

An enterprise produces two articles $V_1, V_2$ and uses three resources $R_1, R_2, R_3$ accessible in [6, 8], [4, 6] and [9, 11] units respectively. We introduce intervals instead of exact values of coefficients to emphasize some problems of consumption of resources.
Table 2.2 shows the consumption of resources to produce one unit of $V_1, V_2$ determined as intervals because of problems with quality and conservation of resources.

To produce one unit of $V_1$ an amount of 2000 SEK and to produce one unit of $V_2$ an amount of 5000 SEK is required. Compute the expected optimal solution $(x, y)$ as numbers of units of $V_1$ and $V_2$ which give the maximal profit of the enterprise.

Table 2.2: The consumption of resources $R_i$ to produce one unit of $V_j$; $i = 1, 2, 3$, $j = 1, 2$

<table>
<thead>
<tr>
<th>$R_i$</th>
<th>$V_1$</th>
<th>$V_2$</th>
<th>Accessibilities</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_1$</td>
<td>[1, 2]</td>
<td>[3, 4]</td>
<td>[6, 8]</td>
</tr>
<tr>
<td>$R_2$</td>
<td>[1, 3]</td>
<td>[2, 5]</td>
<td>[4, 6]</td>
</tr>
<tr>
<td>$R_3$</td>
<td>[2, 4]</td>
<td>[6, 7]</td>
<td>[9, 11]</td>
</tr>
</tbody>
</table>

Solution:

Let $x$ and $y$ represent the units of $V_1$ and $V_2$ respectively. We determine the program

$$\text{Max } P = \text{Max } (2x + 5y)$$

subject to:

$$-x - 3y \leq -6 \quad 2x + 4y \leq 8$$
$$-x - 2y \leq -4 \quad 3x + 5y \leq 6$$
$$-2x - 6y \leq -9 \quad 4x + 7y \leq 11$$
$$x \geq 0, \ y \geq 0.$$

We solve the problem by simplex tabular method.

Let $s_1, s_2, s_3, s_4, s_5, s_6$ be the slack variables and $P$ be the objective value of the optimum solution.
Example 2.4 (Problem of maximization)

A small press shop manufactures 5 models $M_1$, $M_2$, $M_3$, $M_4$, $M_5$ of vehicles. The time required for pressing, grinding and polishing is given in Table 2.3. The manufacturer has 4 presser, 2 grinder and 3 polisher; therefore in 40 hours a week there are 160 pressing hour, 80 grinding hour and 120 polishing hours. The cost to produce each model after each process is evaluated as 20 SEK for $M_1$, 35 SEK for $M_2$, 15 SEK for $M_3$, 10 SEK for $M_4$ and 50 SEK for $M_5$. How many units of each model should be produced to obtain a maximum profit?

Table 2.3: The time required for pressing, grinding and polishing of models $M_i$; $i=1, 2, 3, 4, 5$
Solution:

Let \( v, w, x, y, z \) be the production units of \( M_1, M_2, M_3, M_4 \) and \( M_5 \) respectively. We determine the program

\[
\text{Max } P = \text{Max } (20v + 35w + 15x + 10y + 50z)
\]

subject to:

\[
\begin{align*}
-10v - 12w - 23x - 29y - 85z & \leq -145 \\
-10v - 27w - 18x - 24y - 13z & \leq -75 \\
-20v - 37w - 28x - 39y - 18z & \leq -110 \\
20v + 18w + 27x + 31y + 115z & \leq 175 \\
30v + 33w + 22x + 26y + 16z & \leq 85 \\
30v + 43w + 32x + 41y + 21z & \leq 130
\end{align*}
\]

\( v \geq 0, w \geq 0, x \geq 0, y \geq 0, z \geq 0. \)

We solve the problem by simplex tabular method.

Let \( s_1, s_2, s_3, s_4, s_5, s_6 \) be the slack variables and \( P \) be the objective value of the optimum solution.

Tableau # 1

<table>
<thead>
<tr>
<th>( v )</th>
<th>( w )</th>
<th>( x )</th>
<th>( y )</th>
<th>( z )</th>
<th>( s_1 )</th>
<th>( s_2 )</th>
<th>( s_3 )</th>
<th>( s_4 )</th>
<th>( s_5 )</th>
<th>( s_6 )</th>
<th>( P )</th>
</tr>
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<td>26</td>
<td>16</td>
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<th>( s_5 )</th>
<th>( s_6 )</th>
<th>( P )</th>
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<td>0</td>
<td>10/23</td>
<td>0</td>
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</table>
2.3.2 Algorithm for minimization programs with intervals coefficients in the feasible region

This algorithm is applied to the linear programming problem of the form

$$\text{Min } \sum_{j=1}^{n} c_j x_j = \text{Min } (c_1 x_1 + c_2 x_2 + \cdots + c_n x_n)$$

where $c_j \in \mathbb{R}, j = 1, 2, ..., n$, are the coefficients of the objective function.

We form the restrictions

$$\sum_{j=1}^{n} [a_{ij}, \bar{a}_{ij}] x_j \subseteq [b_i, \bar{b}_i],$$

where $X = (x_1, x_2, ..., x_n)$ are variables of the problem, $a_{ij}$ are the lower borders and $\bar{a}_{ij}$ are the upper borders of the intervals being the coefficients in the left side of the feasible region. The values $b_i, \bar{b}_i$ constitute lower and upper borders for the intervals of the right side of the feasible region restrictions [4, 5, 8].

All upper and lower borders are real numbers. Now we consider the feasible region

$$[a_{11}, \bar{a}_{11}] x_1 + [a_{12}, \bar{a}_{12}] x_2 + \cdots + [a_{1n}, \bar{a}_{1n}] x_n \subseteq [b_1, \bar{b}_1]$$

$$[a_{21}, \bar{a}_{21}] x_1 + [a_{22}, \bar{a}_{22}] x_2 + \cdots + [a_{2n}, \bar{a}_{2n}] x_n \subseteq [b_2, \bar{b}_2]$$

$$...$$

$$[a_{m1}, \bar{a}_{m1}] x_1 + [a_{m2}, \bar{a}_{m2}] x_2 + \cdots + [a_{mn}, \bar{a}_{mn}] x_n \subseteq [b_m, \bar{b}_m]$$

which has the intervals as the coefficients [4, 5, 8].

From above the following program is designed as

$$\text{Min } (c_1 x_1 + c_2 x_2 + \cdots + c_n x_n)$$

$$a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n \leq b_1$$

$$a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n \leq b_2$$

$$...$$

Optimal solution: $P = 128571/988$ SEK; $v = 0, w = 179/90, x = 0, y = 0, z = 719/594$. 

Tableau # 3

<table>
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<td>x</td>
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<td>2/61</td>
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Optimal solution: $P = 128571/988$ SEK; $v = 0, w = 179/90, x = 0, y = 0, z = 719/594$. 

<table>
<thead>
<tr>
<th>P</th>
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</tr>
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<tbody>
<tr>
<td>v</td>
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<td>w</td>
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<td>x</td>
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<td>y</td>
<td>0</td>
</tr>
<tr>
<td>z</td>
<td>719/594</td>
</tr>
</tbody>
</table>
\[ a_{m1} x_1 + a_{m2} x_2 + ... + a_{mn} x_n \leq b_m \]

and

\[ \bar{a}_{11} x_1 + \bar{a}_{12} x_2 + ... + \bar{a}_{1n} x_n \geq \bar{b}_1 \]

\[ \bar{a}_{21} x_1 + \bar{a}_{22} x_2 + ... + \bar{a}_{2n} x_n \geq \bar{b}_2 \]

...

\[ \bar{a}_{m1} x_1 + \bar{a}_{m2} x_2 + ... + \bar{a}_{mn} x_n \geq \bar{b}_m. \]

In that way the feasible region possessing coefficients in the form of intervals was converted to a set of inequalities in which the coefficients are real values. The linear program can be solved by the simplex tabular method.

**Example 2.5 (Problem of minimization)**

Anna has parrots in the cage in her house. She uses two seed mixtures \( A_1 \) and \( A_2 \) to feed them. These cost both 18 SEK per Kg. She knows that three types of nutrients \( N_1, N_2 \) and \( N_3 \) are very useful for the birds, which need 4, 6 and 3 units of \( N_1, N_2 \) and \( N_3 \) respectively per day. Table 2.4 shows how much of each nutrients is included in \( A_1 \) and \( A_2 \).

Table 2.4: The quantity of nutrients \( N_j \) in seed mixtures \( A_i \); \( i = 1, 2, \ j = 1, 2, 3 \)

<table>
<thead>
<tr>
<th>( N_j )</th>
<th>( A_1 )</th>
<th>( A_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( N_1 )</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>( N_2 )</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>( N_3 )</td>
<td>1</td>
<td>3</td>
</tr>
</tbody>
</table>

How much of \( A_1 \) and \( A_2 \) should she buy everyday to meet the parrots needs and to minimize the food cost?

**Solution:**

Let \( x \) and \( y \) represent the cost of the food per day. Then we state the program

\[ \text{Min } P = \text{Min } (18x + 18y) \]

subject to:

\[ 4x + y \geq 4 \]

\[ 3x + 2y \geq 6 \]

\[ x + 3y \geq 3 \]

\[ x \geq 0, \ y \geq 0. \]
As our problem is minimization, so we apply the duality property of max and min programming to convert it into maximization. Therefore the objective function thus becomes

$$\text{Max } Q = \text{Max } (4u + 6v + 3w)$$

subject to:

$$4u + 3v + w \leq 18$$
$$u + 2v + 3w \leq 18$$
$$u \geq 0, v \geq 0, w \geq 0.$$  

Now we apply the inexact programming with intervals as coefficients of a feasible region. We introduce intervals instead of exact values of coefficients because cost can be different in different shops.

Table 2.5: The seed mixtures $A_j$ contain the quantity of nutrients $N_i$; $i = 1, 2, 3, j = 1, 2$

<table>
<thead>
<tr>
<th>$A_j/N_i$</th>
<th>$N_1$</th>
<th>$N_2$</th>
<th>$N_3$</th>
<th>Cost (SEK)</th>
</tr>
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<tr>
<td>$A_1$</td>
<td>[2, 6]</td>
<td>[1, 5]</td>
<td>[0, 3]</td>
<td>[15, 21]</td>
</tr>
<tr>
<td>$A_2$</td>
<td>[0, 3]</td>
<td>[1, 6]</td>
<td>[2, 7]</td>
<td>[16, 20]</td>
</tr>
</tbody>
</table>

$$\text{Max } P = \text{Max } (4u + 6v +3w)$$

subject to:

$$-2u - v \leq -15$$
$$-v - 2w \leq -16$$
$$6u + 5v + 3w \leq 21$$
$$3u + 6v +7w \leq 20$$
$$u \geq 0, v \geq 0, w \geq 0.$$  

We apply simplex tabular method to solve this problem.

Let $s_1, s_2, s_3, s_4$ be the slack variables and $P$ be the objective value of the optimum solution.
Tableau # 1

<table>
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<th></th>
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<th>x_1</th>
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<td>-15</td>
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<td>-2</td>
<td>0</td>
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<td>3</td>
<td>6</td>
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<th>x_2</th>
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<td>7/2</td>
<td>0</td>
<td>-17/6</td>
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<td>13/3</td>
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<td>7/6</td>
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<td>10/3</td>
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Tableau # 3

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<tr>
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<td>0</td>
<td>0</td>
<td>-3/7</td>
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<td>1</td>
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<td>0</td>
</tr>
<tr>
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<td>0</td>
<td>67/21</td>
<td>0</td>
<td>2/7</td>
<td>16/21</td>
<td>1</td>
</tr>
</tbody>
</table>

Optimal solution: \( P = 446/21 \) SEK; \( u = 26/21, v = 19/7, w = 0 \).
Chapter 3

Inexact Programming with $\alpha$-cut forms of Fuzzy Numbers

3.1 Basic Definitions

3.1.1 Set
A set is a collection of elements listed exactly or defined by a common property. e.g. $A$ is a set of natural numbers less than 10

i.e. $A = \{1, 2, 3, ..., 9\}$.

3.1.2 Reliability / Characteristic function
To understand the fuzzy set the concept of reliability or characteristics is very useful. The reliability or the definition of a characteristic function is the function assigned to a set. Let $A$ be the classical set, then we note that the characteristic function is $\mu_A(x)$, $x \in A$. The universe $X$ is a collection of all elements $x$. A is defined by a property or a list $A \subseteq X$.

The function

$$\mu_A : X \rightarrow \{0, 1\}$$

or

$$\mu_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

is called the characteristic function of $A$. We extend the concept of the characteristic function on the fuzzy set in the following definition.

3.1.3 Fuzzy set
Fuzzy set was for the first time defined by Lofti A. Zadeh in 1965 as an extension of classical notion of a set. The word “fuzzy” means “uncertain or imprecise”. If the information is not clearly defined then we introduce fuzziness.

A fuzzy set is a collection of elements which correspond to the definition of ‘$A$’ in the reliability degree equal to 1 or equal to a value belonging to interval $[0, 1]$.

In other words a fuzzy set is described by pairs consisted of the name of an element and the degree with which the element satisfies the definition of the set $A$ [1]. Hence,

$A = \{(x, \mu_A(x), x \in X)\}$

or
A = ∑_{x \in X} \frac{\mu_A(x)}{x}

where the sign ∑ is symbolic and $\mu_A(x)$ for $x \in X$ is called a membership degree of $x$.

**Example 3.1 (Fuzzy set)**

Let $X = \text{“age”} = [0, 100]$. Let $A = \text{“young”}$ be defined by its membership function as

$$
\mu_A(x) = \begin{cases} 
1 & \text{for } 0 \leq x \leq 25 \\
\frac{1}{25}x + 2 & \text{for } 25 \leq x \leq 50 
\end{cases}
$$

We check the value of $x$ at 30 and 40

$\mu_{\text{young}}(30) = -\frac{1}{25}(30) + 2 = 0.8$

$\mu_{\text{young}}(40) = -\frac{1}{25}(40) + 2 = 0.4.$

Figure 3.1 contains the sketch of “young”.

**3.1.4 Support of fuzzy set**

A support of fuzzy set $A$ denoted by $\text{supp}(A)$ is a nonempty set $\text{Supp}(A) = \{ x \in X : \mu_A(x) > 0 \}$.

**Example 3.2 (Support of fuzzy set)**

Let $A = \{ \text{“larger integer in X”} \}$ and $X = \{1, 2, 3, ..., 10\}$. Then set $A$ is determined as

$A = \{0.2/4 + 0.4/5 + 0.6/6 + 0.8/7 + 0.9/8 + 1/9 + 1/10\}$. 
The support of A is given by
\[
\text{Supp}(A) = \{4, 5, 6, 7, 8, 9, 10\}.
\]

### 3.2 Topological operations on fuzzy sets

The topological operations on fuzzy sets are defined as unions and intersections.

#### 3.2.1 Fuzzy topological union

Let \( \tilde{A} = \{(x, \mu_{\tilde{A}}(x), x \in X)\} \) and \( \tilde{B} = \{(x, \mu_{\tilde{B}}(x), x \in X)\} \), then topological union of two fuzzy sets \( \tilde{A} \) and \( \tilde{B} \) is written by \( \tilde{A} \cup \tilde{B} \). It is defined as a fuzzy set \( \tilde{A} \cup \tilde{B} \) with \( \text{Supp}(\tilde{A} \cup \tilde{B}) \) and the membership function
\[
\mu_{\tilde{A} \cup \tilde{B}}(x) = \max_{x \in \text{supp}(\tilde{A} \cup \tilde{B})} (\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)).
\]

**Example 3.3 (Fuzzy topological union)**

Let \( X = \{1, 2, 3, \ldots, 10\} \).

We define
\[
\tilde{A} = \text{“close to 7”} = \{0.2/4 + 0.5/5 + 0.8/6 + 1/7 + 0.8/8 + 0.5/9 + 0.2/10\} \text{ and}
\]
\[
\tilde{B} = \text{“smaller integer in X”} = \{1/1 + 1/2 + 0.8/3 + 0.6/4 + 0.4/5 + 0.2/6\}.
\]

Then supports of \( \tilde{A} \) and \( \tilde{B} \) are sets
\[
\text{Supp}(\tilde{A}) = \{4, 5, 6, 7, 8, 9, 10\} \text{ and}
\]
\[
\text{Supp}(\tilde{B}) = \{1, 2, 3, 4, 5, 6\},
\]

which creates
\[
\text{Supp}(\tilde{A} \cup \tilde{B}) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}.
\]

Hence the union \( \tilde{A} \cup \tilde{B} \) is given by
\[
\tilde{A} \cup \tilde{B} = 1/1 + 1/2 + 0.8/3 + 0.6/4 + 0.5/5 + 0.8/6 + 1/7 + 0.8/8 + 0.5/9 + 0.2/10.
\]

#### 3.2.2 Fuzzy topological intersection

Let \( \tilde{A} = \{(x, \mu_{\tilde{A}}(x), x \in X)\} \) and \( \tilde{B} = \{(x, \mu_{\tilde{B}}(x), x \in X)\} \), then topological intersection of two fuzzy sets \( \tilde{A} \) and \( \tilde{B} \) is denoted by \( \tilde{A} \cap \tilde{B} \). It is defined as a fuzzy set with a \( \text{Supp}(\tilde{A} \cap \tilde{B}) \) and the membership function
\[
\mu_{\tilde{A} \cap \tilde{B}}(x) = \min_{x \in \text{supp}(\tilde{A} \cap \tilde{B})} (\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)).
\]

**Example 3.4 (Fuzzy topological intersection)**

Let \( X = \{1, 2, 3, \ldots, 10\} \).
We define
\[ \tilde{A} = \text{“close to 7”} = \{0.2/4 + 0.5/5 + 0.8/6 + 1/7 + 0.8/8 + 0.5/9 + 0.2/10\} \] and
\[ \tilde{B} = \text{“smaller integer in } X\text{”} = \{1/1 + 1/2 + 0.8/3 + 0.6/4 + 0.4/5 + 0.2/6\}. \]

Then supports of \( \tilde{A} \) and \( \tilde{B} \) are sets
\[ \text{Supp}(\tilde{A}) = \{4, 5, 6, 7, 8, 9, 10\} \] and
\[ \text{Supp}(\tilde{B}) = \{1, 2, 3, 4, 5, 6\}. \]

We determine
\[ \text{Supp}(\tilde{A} \cap \tilde{B}) = \{4, 5, 6\}. \]

Hence the intersection \( \tilde{A} \cap \tilde{B} \) is given by
\[ \tilde{A} \cap \tilde{B} = 0.2/4 + 0.4/5 + 0.2/6. \]

### 3.3 α-cut representation of a fuzzy set

The α-cut of the fuzzy set \( A \) is a nonempty set denoted by \( A_\alpha \) and defined by
\[ A_\alpha = \{x: \mu_A(x) \geq \alpha\}, \quad \alpha \in [0, 1]. \]

#### Example 3.5 (α-cut)

Let \( X = \{1, 2, 3, ..., 10\} \) and \( A = \text{“close to 5”}. \)

If \( A \) is “exactly 5” then the crisp set is 1/5.

We define
\[ A = \text{“close to 5”} = \{0.2/2 + 0.5/3 + 0.8/4 + 1/5 + 0.8/6 + 0.5/7 + 0.2/8\}. \]

For \( \alpha = 0.5, 0.8 \)
\[ A_{0.5} = \{3, 4, 5, 6, 7\} \quad \text{and} \quad A_{0.8} = \{4, 5, 6\}. \]

#### 3.3.1 Normal fuzzy set

A fuzzy set is called normal if at least one element \( x \in A \) has a membership degree equal to 1.

#### 3.3.2 Cardinality of a fuzzy set

If \( A = \{(x, \mu_A(x), x \in X)\} \) is a fuzzy set then cardinality of \( A \) denoted by \( |A| \) is equal to
\[ |A| = \sum_{x \in X} \mu_A(x). \]
Example 3.6 (Cardinality of fuzzy set)

If $X = \{0, 1, 2, \ldots, 10\}$ and

$A = \text{"close to 5"} = \{0.2/2 + 0.5/3 + 0.8/4 + 1/5 + 0.8/6 + 0.5/7 + 0.2/8\}$,

then cardinality of fuzzy set $A$ is

$|A| = 0.2 + 0.5 + 0.8 + 1 + 0.8 + 0.5 + 0.2 = 4$.

The related cardinality of fuzzy set $A$ is defined as

$\|A\| = \frac{|A|}{|X|}$ and is practically calculated, e.g. for $A$ from Ex.3.6 as

$\|A\| = \frac{4}{11} = 0.35$.

3.4 Fuzzy number

1. A fuzzy number is a special fuzzy set defined in the space of real numbers $R$.
2. A fuzzy number must have at least one element $x_0$ with membership degree equal to 1.
3. The membership in the fuzzy number must grow to $x_0$ and decrease from $x_0$.

Example 3.7 (Fuzzy number)

Let $X = \{1, 2, 3, \ldots, 10\}$ and $A = \text{"close to 5"}$.

Clearly $A$ is a fuzzy number, since it satisfies all the three properties of the definition of fuzzy number. Figure 3.2 presents the graph of $A$. 

![Figure 3.2 "close to 5"](image-url)
3.4.1 Zero fuzzy number
A zero fuzzy number \( A \) must have 0 included in its support \( (0 \in \text{supp}(A)) \).

3.4.2 Fuzzy number in L-R form
Let \( A \) be a fuzzy number in L-R form having a membership function \( \mu_A(x) \), which refers to two functions L & R and two scalars \( \alpha \& \beta \) in the following way [3].

If \( A = (m_A, \alpha_A, \beta_A)_{LR} \) then

\[
\mu_A(x) = \begin{cases} 
L \left( \frac{m_A-x}{\alpha_A} \right) & m_A - \alpha_A \leq x \leq m_A \\
R \left( \frac{x-m_A}{\beta_A} \right) & m_A \leq x \leq m_A + \beta_A 
\end{cases}
\]

where \( m_A \) is the mean value, \( \alpha_A \) is the left spread and \( \beta_A \) is the right spread of the L-R type fuzzy number. Figure 3.3 shows the parameters of \( A \).

Figure 3.3 Fuzzy number in L-R form

3.4.3 Interval form of fuzzy number
We can find interval form of a fuzzy number \( A \) from the L-R form by the operation

\[
A = (m_A, \alpha_A, \beta_A)_{LR} = [m_A - \alpha_A, m_A, m_A + \beta_A] = [a_A, m_A, b_A]_{int} .
\]

The membership function of \( A_{int} \)

\[
\mu_A(x) = \begin{cases} 
\frac{x-a_A}{m_A-a_A} & a_A \leq x \leq m_A \\
\frac{b_A-x}{b_A-m_A} & m_A \leq x \leq b_A
\end{cases}
\]
Figure 3.4 shows the parameters of the interval form of a fuzzy number $A$.

**Example 3.8 (Interval form of fuzzy number)**

$A$ = “nearly 50” = $(50, 5, 4)_L$, $R = [45, 50, 54]_I$

has membership function

$$
\mu_A(x) = \begin{cases} 
\frac{x-45}{50-45} & 45 \leq x \leq 50 \\
\frac{54-x}{54-50} & 50 \leq x \leq 54.
\end{cases}
$$

Figure 3.5 shows the parameters of the interval form of a fuzzy number $A$. 

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3.5 \(\alpha\)-cut representations of fuzzy numbers

A fuzzy number \(A\) is represented in the \(\alpha\)-cut form as an interval with borders being two reference functions. Generally, the \(\alpha\)-cut form \(A_\alpha\) of a fuzzy number is given by

\[ A_\alpha = \{x : \mu_A(x) \geq \alpha\}, \quad \alpha \in [0, 1]. \]

We denote \(A\) in the \(\alpha\)-cut form

\[ A(\alpha) = [a(\alpha), b(\alpha)], \quad \alpha \in [0, 1]. \]

The left function \(a(\alpha)\) is estimated by the expression

\[ a(\alpha) = a_A + (m_A - a_A) \alpha \]

and the right function \(b(\alpha)\) is determined as

\[ b(\alpha) = b_A - (b_A - m_A) \alpha. \]

3.6 Operations on fuzzy numbers in the \(\alpha\)-cut form

For two fuzzy numbers \(A\) and \(B\) given as

\[ A(\alpha) = [a_A(\alpha), b_A(\alpha)], B(\alpha) = [a_B(\alpha), b_B(\alpha)] \]

we define the following operations.

3.6.1 Addition of fuzzy numbers in the \(\alpha\)-cut form

Let \(A\) and \(B\) be two fuzzy numbers in the \(\alpha\)-cut form. We define addition as

\[ (A + B)(\alpha) = [a_A(\alpha) + a_B(\alpha), b_A(\alpha) + b_B(\alpha)]. \]

**Example 3.9 (Addition of fuzzy numbers in the \(\alpha\)-cut form)**

Let \(A\) and \(B\) be the two fuzzy numbers in the \(\alpha\)-cut form

\[ A(\alpha) = (2 + 3\alpha, 7 - 2\alpha) \]

\[ B(\alpha) = (1 + 3\alpha, 5 - \alpha). \]

Then the addition result is given by

\[ (A + B)(\alpha) = (2 + 3\alpha + 1 + 3\alpha, 7 - 2\alpha + 5 - \alpha) \]

\[ = (3 + 6\alpha, 12 - 3\alpha). \]

3.6.2 Subtraction of fuzzy numbers in the \(\alpha\)-cut form

Let \(A\) and \(B\) be two fuzzy numbers in the \(\alpha\)-cut form. We define subtraction as

\[ (A - B)(\alpha) = [a_A(\alpha) - b_B(\alpha), b_A(\alpha) - a_B(\alpha)]. \]
Example 3.10 (Subtraction of fuzzy numbers in the $\alpha$-cut form)

Let $A$ and $B$ be the two fuzzy numbers in the $\alpha$-cut form

$A(\alpha) = (2 + 3\alpha, 7 - 2\alpha)$ and

$B(\alpha) = (1 + 3\alpha, 5 - \alpha)$.

Then the subtraction result is given by

$(A - B)(\alpha) = (2 + 3\alpha - (5 - \alpha), 7 - 2\alpha - (1 + 3\alpha))$

$= (-3 + 4\alpha, 6 - 5\alpha)$.

3.6.3 Multiplication of fuzzy numbers in the $\alpha$-cut form

Let $A$ and $B$ be two fuzzy numbers in the $\alpha$-cut form. We define multiplication as

$(A \cdot B)(\alpha) = [a_A(\alpha) \cdot a_B(\alpha), b_A(\alpha) \cdot b_B(\alpha)]$.

Example 3.11 (Multiplication of fuzzy numbers in the $\alpha$-cut form)

Let $A$ and $B$ be the two fuzzy numbers in the $\alpha$-cut form

$A(\alpha) = (1 + \alpha, 4 - 2\alpha)$ and

$B(\alpha) = (3 + 2\alpha, 7 - 2\alpha)$.

Then the multiplication result is given by

$(A \cdot B)(\alpha) = ((1 + \alpha)(3 + 2\alpha), (4 - 2\alpha)(7 - 2\alpha))$

$= (3 + 5\alpha + 2\alpha^2, 28 - 22\alpha + 4\alpha^2)$.

3.6.4 Division of fuzzy numbers in the $\alpha$-cut form

Let $A$ and $B$ be two fuzzy numbers in the $\alpha$-cut form. We define division as

$(A \div B)(\alpha) = [a_A(\alpha) \div b_B(\alpha), b_A(\alpha) \div a_B(\alpha)]$.

Example 3.12 (Division of fuzzy numbers in the $\alpha$-cut form)

Let $A$ and $B$ be the two fuzzy numbers in the $\alpha$-cut form

$A(\alpha) = (2 + 3\alpha, 7 - 2\alpha)$ and

$B(\alpha) = (1 + 3\alpha, 5 - \alpha)$.

Then the division result is given by

$(A \div B)(\alpha) = ((2 + 3\alpha)/(5 - \alpha), (7 - 2\alpha)/(1 + 3\alpha))$. 
3.7 Inexact programming with α-cut forms of fuzzy numbers in the feasible region of LPP (linear programming problems)

3.7.1 Algorithm for maximization with α-cut forms of fuzzy numbers

This algorithm concerns the linear objective function of the form

$$ \text{Max} \sum_{j=1}^{n} c_j x_j = \text{Max} \left( c_1 x_1 + c_2 x_2 + \cdots + c_n x_n \right) $$

with $c_j \in \mathbb{R}$, $j = 1, 2, \ldots, n$, which are the coefficients of an objective function followed by the constraints

$$ x_1 A_1 \cup x_2 A_2 \cup \ldots \cup x_n A_n \subseteq A $$

where $X = (x_1, x_2, \ldots, x_n)$ are variables of the problem and $A_i$, $i = 1, 2, \ldots, n$, are the fuzzy numbers in $\mathbb{R}$ with

$$\mu_{A_i} : \mathbb{R} \to [0, 1] \text{ and } \mu_A : \mathbb{R} \to [0, 1].$$

The inclusion of fuzzy sets $A \subseteq B$ is defined by

$$\mu_A(x) \leq \mu_B(x) \text{ for each } x \in X.$$ 

$A_1 \cup A_2$ has $\mu_{A_1 + A_2}(z) = \max z = x + y \left( \min(\mu_{A_1}(x), \mu_{A_2}(x)) \right), x \in \mathbb{R}.$

$kA_i$ has $\mu_{kA_i}(x) = k \mu_{A_i}(x) ; x \in X.$

$A = \bigcup_{\alpha \in [0, 1]} \alpha A_\alpha$ where $\alpha A_\alpha$ is a fuzzy set, which has elements equal to elements from $A_\alpha$ and all memberships equal to $\alpha$. Thus [4, 5, 8]

$$\text{Max} \left( c_1 x_1 + c_2 x_2 + \cdots + c_n x_n \right) \text{ is followed by }$$

$$x_1 A_{1\alpha} \cup x_2 A_{2\alpha} \cup \ldots \cup A_{n\alpha} \subseteq A_\alpha \text{ for all } \alpha \in [0, 1].$$

If $\alpha = \{r_1, r_2, \ldots, r_p\}$

then $r_1, r_2, \ldots, r_p$ are chosen values of $\alpha$ to create $\alpha$-cuts $A_{1r_1}, A_{1r_2}$ and so on.

It is proved that the program is converted to the form [4]

$$\text{Max} \left( c_1 x_1 + c_2 x_2 + \cdots + c_n x_n \right)$$

$$x_1 A_{1r_1} + x_2 A_{2r_1} + \cdots + x_n A_{nr_1} \subseteq A_{r_1}$$

$$\ldots$$

$$x_1 A_{1r_p} + x_2 A_{2r_p} + \cdots + x_n A_{nr_p} \subseteq A_{r_p}.$$
Thus the problem has now intervals as coefficients in the feasible region and it can be solved as the inexact programming with intervals.

Example 3.13

An enterprise produces two articles $V_1, V_2$ and uses one resource $R_1$ accessible in “close to 7” units, because of some uncertainty in delivery of $R_1$. 

Table 3.1 shows the consumption of the resource to produce one unit of $V_1, V_2$, where the consumed amounts of $R_1$ are determined as fuzzy numbers because of problems with quality and conservation of resource $R_1$.

To produce one unit of $V_1$ costs 2 SEK and to produce one unit of $V_2$ costs 5 SEK. Compute the expected optimal solution $(x, y)$ as numbers of units of $V_1$ and $V_2$ which give the maximal profit of the enterprise.

Table 3.1: The consumption of resource $R_1$ for production of $V_1$ and $V_2$ respectively

<table>
<thead>
<tr>
<th>$R_1$</th>
<th>$V_j$</th>
<th>$V_1$</th>
<th>$V_2$</th>
<th>accessibilities</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_1$</td>
<td>Close to 1 $= A_1$</td>
<td>Close to 3 $= A_2$</td>
<td>Close to 7 $= A$</td>
<td></td>
</tr>
</tbody>
</table>

Solution:

For the fuzzy numbers $A_1, A_2$ and $A$ showing consumption of $R_1$ and its access respectively, we decide the $\alpha$-cuts for $\alpha = 0, 0.5$ and 1 as shown in figure 3.6 to build a feasible region.

![Figure 3.6 α-cuts for 0, 0.5, and 1](image-url)
\[ \alpha = r_1 = 0 \quad x[0, 2] + y[2, 4] \subseteq [6, 8], \]
\[ \alpha = r_2 = 0.5 \quad x[0.5, 1.5] + y[2.5, 3.5] \subseteq [6.5, 7.5], \]
\[ \alpha = r_3 = 1 \quad x + 3y = 7. \]

We consider the program

\[ \text{Max } P = \text{Max } (2x + 5y) \]

subject to:

\[ x + 3y \leq 7 \]
\[ -x - 3y \leq -7 \]
\[ -(1/2)x - (5/2)y \leq -(13/2) \]
\[ (3/2)x + (7/2)y \leq (25/2) \]
\[ -2y \leq -6 \]
\[ 2x + 4y \leq 8 \]
\[ x, y \geq 0. \]

We solve the problem by simplex tabular method.

Let \( s_1, s_2, s_3, s_4, s_5, s_6 \) be the slack variables and \( P \) be the objective value of the optimum solution.

Tableau # 1

<table>
<thead>
<tr>
<th></th>
<th>x</th>
<th>y</th>
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<th>( s_2 )</th>
<th>( s_3 )</th>
<th>( s_4 )</th>
<th>( s_5 )</th>
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<th>$R_4$</th>
<th>$R_5$</th>
<th>P</th>
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<td>0</td>
<td>-7/8</td>
</tr>
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<td>0</td>
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</tr>
</tbody>
</table>

Optimal solution: $P = 10$ SEK; $x = 0, y = 2$.

Example 3.14

An enterprise produces two articles $V_1, V_2$ and uses two resources $R_1, R_2$ accessible in “close to 7” and “close to 5” units respectively. We evaluate the imprecise amounts of resources because of some problems in their delivery.

Table 3.2 shows the consumption of resources to produce one unit of $V_1, V_2$ determined as fuzzy numbers.

To produce one unit of $V_1$ costs 2 SEK and to produce one unit of $V_2$ costs 5 SEK. Compute the expected optimal solution $(x, y)$ as a pair of numbers of units of $V_1$ and $V_2$, which give the maximal profit of the enterprise.

Table 3.2: The consumption of resources $R_i$ to produce one unit of $V_j$, $i = 1, 2$, $j = 1, 2$

<table>
<thead>
<tr>
<th>$R_i$</th>
<th>$V_1$</th>
<th>$V_2$</th>
<th>Access to resources</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_1$</td>
<td>Close to 1</td>
<td>Close to 3</td>
<td>Close to 7</td>
</tr>
<tr>
<td>$R_2$</td>
<td>Close to 2</td>
<td>Close to 3.5</td>
<td>Close to 5</td>
</tr>
</tbody>
</table>

Solution:

For the fuzzy numbers, showing consumption of $R_1$ and its access we decide the $\alpha$-cuts for $\alpha = 0, 0.5$ and 1 to be included in a set constituting the first part of the feasible region (Fig.3.6)

$\alpha = r_1 = 0$ \quad $x[0, 2] + y[2,4] \subseteq [6,8],$

$\alpha = r_2 = 0.5$ \quad $x[0.5, 1.5] + y[2.5, 3.5] \subseteq [6.5, 7.5],$

$\alpha = r_3 = 1$ \quad $x + 3y = 7.$

For the fuzzy numbers, showing consumption of $R_2$ and its access we decide the $\alpha$-cuts for $\alpha = 0, 0.5$ and 1 as shown in figure 3.7 to obtain the second part of the feasible region as
\[\alpha = r_1 = 0 \quad x[1, 3] + y[2.5, 4.5] \subseteq [4, 6],\]
\[\alpha = r_2 = 0.5 \quad x[1.5, 2.5] + y[3, 4] \subseteq [4.5, 5.5],\]
\[\alpha = r_3 = 1 \quad 2x + 3.5y = 5.\]

The program is finally stated in the form of

\[
\text{Max } P = \text{Max } (2x + 5y)
\]

subject to:

\[
\begin{align*}
x + 3y & \leq 7 \\
-x - 3y & \leq -7 \\
-(1/2)x - (5/2)y & \leq -(13/2) \\
(3/2)x + (7/2)y & \leq (25/2) \\
-2y & \leq -6 \\
2x + 4y & \leq 8 \\
2x + (7/2)y & \leq 5 \\
-2x - (7/2)y & \leq -5 \\
-(3/2)x - 3y & \leq -(9/2) \\
(5/2)x + 4y & \leq (11/2)
\end{align*}
\]
\[-x - \frac{5}{2}y \leq -4\]
\[3x + \frac{9}{2}y \leq 6\]
\[x, y \geq 0.\]

We solve the problem by simplex tabular method.

Let \(s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8, s_9, s_{10}, s_{11}, s_{12}\) be the slack variables and \(P\) be the objective value of the optimum solution.

Tableau # 1

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<thead>
<tr>
<th>x</th>
<th>y</th>
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<th>(s_2)</th>
<th>(s_3)</th>
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Optimal solution: \( P = \frac{20}{3} \) SEK; \( x = 0, y = \frac{4}{3} \).
Chapter 4

**L-R Representations of Fuzzy Numbers in Inexact Programming**

Let us return to definition (3.4.2 chapter 3), which implements a fuzzy number in the L-R form. To explain its action even better we provide example 4.1.

**Example 4.1 (Fuzzy number in L-R form)**

Let $A = (5, 3, 2)_{LR}$, then the membership function for $A$ is given by

$$
\mu_A(x) = \begin{cases} 
L\left(\frac{5-x}{3}\right) & 2 \leq x \leq 5 \\
R\left(\frac{x-5}{2}\right) & 5 \leq x \leq 7.
\end{cases}
$$

If $(x) = R(x) = 1 - x$, then we can find left and right parts of $A$’s membership function as

$$
L\left(\frac{5-x}{3}\right) = 1 - \frac{5-x}{3} = \frac{x-2}{3}
$$

$$
R\left(\frac{x-5}{2}\right) = 1 - \frac{x-5}{2} = \frac{7-x}{2}.
$$

The required membership function of $A$ is thus given by

$$
\mu_A(x) = \begin{cases} 
\frac{x-2}{3} & 2 \leq x \leq 5 \\
\frac{7-x}{2} & 5 \leq x \leq 7.
\end{cases}
$$

### 4.1 Operations on fuzzy numbers in LR form

For two fuzzy numbers $A$ and $B$ in L-R form given as $A = (m_A, a_A, b_A)_{LR}$, $B = (m_B, a_B, b_B)_{LR}$

we can define the following operations [3].

#### 4.1.1 Addition of fuzzy numbers in LR form

Let $A$ and $B$ be two fuzzy numbers in L-R form. We define addition as

$$(A + B)_{LR} = (m_A, a_A, b_A)_{LR} + (m_B, a_B, b_B)_{LR}$$

$$=(m_A + m_B, a_A + a_B, b_A + b_B)_{LR}.$$
Example 4.2 (Addition of fuzzy numbers in LR form)

Let $A = (4, 1, 3)_{LR}$ and $B = (6, 2, 1)_{LR}$ be two fuzzy numbers in L-R form.

Then the addition result is given by

$$(A + B)_{LR} = (4+6, 1+2, 3+1) = (10, 3, 4)_{LR}.$$ 

4.1.2 Subtraction of fuzzy numbers in LR form

Let $A$ and $B$ be two fuzzy numbers in L-R form. We define subtraction as

$$(A - B)_{LR} = (m_A, \alpha_A, \beta_A)_{LR} - (m_B, \alpha_B, \beta_B)_{RL}$$

$$= (m_A - m_B, \alpha_A + \beta_B, \beta_A + \alpha_B)_{RL}.$$ 

Example 4.3 (Subtraction of fuzzy numbers in LR form)

Let $A = (4, 1, 3)_{LR}$ and $B = (6, 2, 1)_{LR}$ be two fuzzy numbers in L-R form.

Then the subtraction result is given by

$$(A - B)_{LR} = (4-6, 1+1, 3+2) = (-2, 2, 5)_{RL}.$$ 

4.1.3 Multiplication of fuzzy numbers in LR form

Let $A$ and $B$ be two fuzzy numbers in L-R form. Then the multiplication

$$(A \cdot B)_{LR} = (m_A, \alpha_A, \beta_A)_{LR} \cdot (m_B, \alpha_B, \beta_B)_{LR}$$

is defined by the following three ways depending on mean values of the numbers.

I. For $m_A > 0$, $m_B > 0$, then

$$A \cdot B_{LR} \simeq (m_A m_B, m_A \alpha_B + m_B \alpha_A, m_A \beta_B + m_B \beta_A)_{LR}.$$ 

II. For $m_A < 0$, $m_B > 0$, then

$$A \cdot B_{LR} \simeq (m_A m_B, m_B \alpha_A - m_A \beta_B, m_B \beta_A - m_A \alpha_B)_{RL}.$$ 

III. For $m_A < 0$, $m_B < 0$, then

$$A \cdot B_{LR} \simeq (m_A m_B, -m_B \beta_A - m_A \beta_B, -m_B \alpha_A - m_A \alpha_B)_{RL}.$$ 

Example 4.4 (Multiplication of fuzzy numbers in LR form)

I. Let $A = (4, 1, 3)_{LR}$ and $B = (6, 2, 1)_{LR}$ be two fuzzy numbers in L-R form for the first case.
Then the multiplication result is given by

\[(A \cdot B)_{LR} \approx (24, 8+6, 4+18)\]

\[= (24, 14, 22)_{LR}.
\]

**II.** Let \(A = (-4, 1, 3)_{LR}\) and \(B = (6, 2, 1)_{LR}\) be two fuzzy numbers in L-R form for the second case.

Then the multiplication result is given by

\[(A \cdot B)_{LR} \approx (-24, 6+4, 18+8)\]

\[= (-24, 10, 26)_{RL}.
\]

**III.** Let \(A = (-4, 1, 3)_{LR}\) and \(B = (-6, 2, 1)_{LR}\) be two fuzzy numbers in L-R form for the third case.

Then the multiplication result is given by

\[(A \cdot B)_{LR} \approx (-24, 18+4, 6+8)\]

\[= (-24, 22, 14)_{RL}.
\]

**4.1.4 Division of fuzzy numbers in LR form**

Let \(A\) and \(B\) be two fuzzy numbers in L-R form. Then the division

\[(A \div B)_{LR} = (m_A, \alpha_A, \beta_A)_{LR} \div (m_B, \alpha_B, \beta_B)_{RL}\]

is depending on mean values of the numbers and is defined as.

If \(m_A > 0, m_B > 0\), then we can find the inverse of \(B\) as

\[B^{-1} = (m_B, \alpha_B, \beta_B)_{LR}^{-1}\]

\[\approx \left(\frac{1}{m_B}, \frac{\beta_B}{m_B}, \frac{\alpha_B}{m_B}\right)_{RL},\text{ so the division is given by}\]

\[(A \div B)_{LR} \approx \left(\frac{m_A}{m_B}, \frac{m_B \alpha_A + m_A \beta_B}{m_B^2}, \frac{m_A \alpha_B + m_B \beta_A}{m_B^2}\right)_{LR}.
\]

**Example 4.5 (Division of fuzzy numbers in LR form)**

Let \(A = (4, 1, 3)_{LR}\) and \(B = (6, 2, 1)_{LR}\) be two fuzzy numbers in L-R form.

Since \(6 > 0\), so we can find \(B^{-1}\) as

\[B^{-1} \approx \left(\frac{1}{6}, \frac{1}{36}, \frac{2}{36}\right)_{LR}.
\]
Then the division result is given by

\[
(A \div B)_{LR} \simeq \left( \frac{4}{6} \right)^{\frac{6+4}{36}, \frac{8+18}{36}}
= \left( \frac{2}{3}, \frac{10}{36}, \frac{26}{36} \right)
= \left( \frac{2}{3}, \frac{5}{18}, \frac{13}{18} \right)_{LR}.
\]

4.2 Order of fuzzy numbers in LR form

The order for two fuzzy numbers in L-R form can be defined by means of min and max operations [3].

4.2.1 Maximum of fuzzy numbers in LR form

Let A and B be two fuzzy numbers in L-R form.

If \( A = (m_A, \alpha_A, \beta_A)_{LR} \) and \( B = (m_B, \alpha_B, \beta_B)_{LR} \), then the order or rank of two fuzzy numbers A and B is defined by the following two ways

I. If \( m_A \gg m_B \) then max(A, B) is defined as

\[
\max (A, B) = \max ((m_A, \alpha_A, \beta_A)_{LR}, (m_B, \alpha_B, \beta_B)_{LR}) \simeq (m_A, \alpha_A, \beta_A)_{LR}.
\]

In this case, \( m_A \gg m_B \), which means that there is no intersection point between A and B i.e. \( \text{supp}(A) \cap \text{supp}(B) = \emptyset \).

And if

II. \( m_A \) is close to \( m_B \), then max(A, B) is defined as

\[
\max (A, B) = \max ((m_A, \alpha_A, \beta_A)_{LR}, (m_B, \alpha_B, \beta_B)_{LR})
\simeq (\max (m_A, m_B), \min(\alpha_A, \alpha_B), \max (\beta_A, \beta_B))_{LR}.
\]

In this case, \( m_A \) is close to \( m_B \), which means that there are any intersection points between A and B i.e. \( \text{supp}(A) \cap \text{supp}(B) \neq \emptyset \).

Example 4.6 (Maximum of fuzzy numbers in LR form)

I. Let \( A = (4, 0, 4, 5)_{LR} \) and \( B = (4, 1, 3)_{LR} \) be two fuzzy numbers in L-R form.

Since \( m_A \gg m_B \) then, according to 4.3.1-I, maximum is given by

\[
\max (A, B) = \max ((4, 0, 4, 5)_{LR}, (4, 1, 3)_{LR})
\simeq (40, 4, 5)_{LR}.
\]
II. Let \( A = (6, 2, 1)_{LR} \) and \( B = (4, 1, 3)_{LR} \) be two fuzzy numbers in L-R form. Since \( m_A \) is close to \( m_B \) then, according to 4.3.1-II, maximum is given by 
\[
\max((6, 2, 1)_{LR}, (4, 1, 3)_{LR}) \simeq (\max(6, 4), \min(2, 1), \max(1, 3))
\]
\[
= (6, 1, 3)_{LR}.
\]

4.2.2 Minimum of fuzzy numbers in LR form
Let A and B be two fuzzy numbers in L-R form.

If \( A = (m_A, \alpha_A, \beta_A)_{LR} \) and \( B = (m_B, \alpha_B, \beta_B)_{LR} \), then the order or rank of two fuzzy numbers A and B is defined by the following two ways

I. If \( m_A \gg m_B \), then \( \min (A, B) \) is defined as
\[
\min (A, B) = \min (((m_A, \alpha_A, \beta_A)_{LR}, (m_B, \alpha_B, \beta_B)_{LR}) \simeq (m_A, \alpha_A, \beta_A)_{LR}.
\]
In this case \( m_A \gg m_B \) which means that there is no intersection point between A and B i.e. \( \text{supp}(A) \cap \text{supp}(B) = \emptyset \).

And if

II. \( m_A \) is close to \( m_B \), then the \( \min(A, B) \) is defined as
\[
\min (A, B) = \min (((m_A, \alpha_A, \beta_A)_{LR}, (m_B, \alpha_B, \beta_B)_{LR}) \simeq (\min(m_A, m_B), \max(\alpha_A, \alpha_B), \min(\beta_A, \beta_B))_{LR}.
\]
In this case, \( m_A \) is close to \( m_B \) which means that there are any intersection points between A and B i.e. \( \text{supp}(A) \cap \text{supp}(B) \neq \emptyset \).

Example 4.7 (Minimum of fuzzy numbers in LR form)

I. Let \( A = (40, 4, 5)_{LR} \) and \( B = (4, 1, 3)_{LR} \) be two fuzzy numbers in L-R form.

Since \( m_A \gg m_B \) then, according to 4.3.2-I, minimum is given by
\[
\min (A, B) = \min((40, 4, 5)_{LR}, (4, 1, 3)_{LR})
\]
\[
\simeq (4, 1, 3)_{LR}.
\]

II. Let \( A = (6, 2, 1)_{LR} \) and \( B = (4, 1, 3)_{LR} \) be two fuzzy numbers in L-R form.

Since \( m_A \) is close to \( m_B \) then, according to 4.3.2-II, minimum is given by
\[
\min ((6, 2, 1)_{LR}, (4, 1, 3)_{LR}) \simeq (\min(6, 4), \max(2, 1), \min(1, 3))
\]
\[
= (4, 2, 1)_{LR}.
\]
4.3 Inexact programming with L-R forms of fuzzy numbers in the feasible region of LPP (linear programming problems)

4.3.1 Algorithm for maximization with L-R forms of fuzzy numbers

This algorithm is applied to the linear programming problems of the form

\[ \text{Max} \sum_{j=1}^{n} c_j x_j = \text{Max} (c_1 x_1 + c_2 x_2 + \cdots + c_n x_n) \]

subject to:

\[ A_{11} x_1 + A_{12} x_2 + \cdots + A_{1n} x_n \leq b_1 \]
\[ A_{21} x_1 + A_{22} x_2 + \cdots + A_{2n} x_n \leq b_2 \]
\[ \vdots \]
\[ A_{m1} x_1 + A_{m2} x_2 + \cdots + A_{mn} x_n \leq b_m, \]

where \( c_j \in \mathbb{R}; j = 1, 2, 3, \ldots, n \) are the coefficients of objective functions. \( X = (x_1, x_2, \ldots, x_n) \) are variables of the problem. All the coefficients \( A_{ij}, i = 1, 2, \ldots, m; j = 1, 2, \ldots, n \) of the feasible region are in the L-R form of fuzzy numbers. The right sides of the feasible region restrictions \( b_1, b_2, \ldots, b_m \) are in the L-R form of the fuzzy numbers.

Now we consider the feasible region, which has all the coefficients in L-R form, as a set of inequalities

\[ (m_{A_{11}}, a_{A_{11}}, b_{A_{11}}) x_1 + (m_{A_{12}}, a_{A_{12}}, b_{A_{12}}) x_2 + \cdots + (m_{A_{1n}}, a_{A_{1n}}, b_{A_{1n}}) x_n \leq (m_{b_1}, a_{b_1}, b_{b_1}) \]
\[ (m_{A_{21}}, a_{A_{21}}, b_{A_{21}}) x_1 + (m_{A_{22}}, a_{A_{22}}, b_{A_{22}}) x_2 + \cdots + (m_{A_{2n}}, a_{A_{2n}}, b_{A_{2n}}) x_n \leq (m_{b_2}, a_{b_2}, b_{b_2}) \]
\[ \vdots \]
\[ (m_{A_{m1}}, a_{A_{m1}}, b_{A_{m1}}) x_1 + (m_{A_{m2}}, a_{A_{m2}}, b_{A_{m2}}) x_2 + \cdots + (m_{A_{mn}}, a_{A_{mn}}, b_{A_{mn}}) x_n \leq (m_{b_m}, a_{b_m}, b_{b_m}). \]

When performing the multiplications of fuzzy numbers by constants and additions of products we convert set of inequalities with coefficients in L-R form, to the form

\[ (m_{A_{11}} x_1 + m_{A_{12}} x_2 + \cdots + m_{A_{1n}} x_n, a_{A_{11}} x_1 + a_{A_{12}} x_2 + \cdots + a_{A_{1n}} x_n, b_{A_{11}} x_1 + b_{A_{12}} x_2 + \cdots + b_{A_{1n}} x_n) \leq (m_{b_1}, a_{b_1}, b_{b_1}) \]
\[ (m_{A_{21}} x_1 + m_{A_{22}} x_2 + \cdots + m_{A_{2n}} x_n, a_{A_{21}} x_1 + a_{A_{22}} x_2 + \cdots + a_{A_{2n}} x_n, b_{A_{21}} x_1 + b_{A_{22}} x_2 + \cdots + b_{A_{2n}} x_n) \leq (m_{b_2}, a_{b_2}, b_{b_2}) \]
\[ \vdots \]
Now each inequality presenting the fuzzy numbers in L-R form will be split in a set of three inequalities with real coefficients which is proved by [6] in compliance with order operations defined by subsections 4.2.1 and 4.2.2

\[ m_{A_{11}}x_1 + m_{A_{12}}x_2 + \ldots + m_{A_{1n}}x_n \leq m_{b_1} \]
\[ \alpha_{A_{11}}x_1 + \alpha_{A_{12}}x_2 + \ldots + \alpha_{A_{1n}}x_n \leq \alpha_{b_1} \]
\[ \beta_{A_{11}}x_1 + \beta_{A_{12}}x_2 + \ldots + \beta_{A_{1n}}x_n \leq \beta_{b_1}, \]

and

\[ m_{A_{21}}x_1 + m_{A_{22}}x_2 + \ldots + m_{A_{2n}}x_n \leq m_{b_2} \]
\[ \alpha_{A_{21}}x_1 + \alpha_{A_{22}}x_2 + \ldots + \alpha_{A_{2n}}x_n \leq \alpha_{b_2} \]
\[ \beta_{A_{21}}x_1 + \beta_{A_{22}}x_2 + \ldots + \beta_{A_{2n}}x_n \leq \beta_{b_2} \]

\[ \ldots \]
\[ m_{A_{m1}}x_1 + m_{A_{m2}}x_2 + \ldots + m_{A_{mn}}x_n \leq m_{b_m} \]
\[ \alpha_{A_{m1}}x_1 + \alpha_{A_{m2}}x_2 + \ldots + \alpha_{A_{mn}}x_n \leq \alpha_{b_m} \]
\[ \beta_{A_{m1}}x_1 + \beta_{A_{m2}}x_2 + \ldots + \beta_{A_{mn}}x_n \leq \beta_{b_m}. \]

In that way we converted the feasible region with coefficients being L-R fuzzy numbers into a region containing real numbers as new coefficients. Hence the linear program can be solved by the simplex tabular method.

**Example 4.8**

Let us consider a linear program with L-R fuzzy numbers in the feasible region

\[ \text{Max } Z = \text{Max } (6x + 5y) \]

subject to:

\[ (5, 3, 2)_{LR} x + (6, 4, 2)_{LR} y \leq (25, 6, 9)_{LR} \]
\[ (5, 2, 3)_{LR} x + (2, 1.5, 1)_{LR} y \leq (13, 7, 4)_{LR} \]
\[ x \geq 0, y \geq 0. \]
Solution:

Let $x$ and $y$ represent the variables of the problem. Then we state the program

$$\text{Max } Z = \text{Max } (6x + 5y)$$

subject to:

$$5x + 6y \leq 25$$
$$3x + 4y \leq 6$$
$$2x + 2y \leq 9$$
$$5x + 2y \leq 13$$
$$2x + 1.5y \leq 7$$
$$3x + y \leq 4$$

$x \geq 0, y \geq 0.$

We solve the problem by simplex tabular method.

Let $s_1, s_2, s_3, s_4, s_5, s_6$ be the slack variables and $Z$ be the objective value of the optimum solution.

Tableau # 1

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Optimal solution: $z = 10; x = 10/9, y = 2/3$.

We emphasize that the action of programs including L-R forms of fuzzy numbers in the feasible region substantially reduces the number of restrictions with real numbers. Every fuzzy restriction inequality is assisted by only three inequalities, when comparing to programs containing α-cut in the feasible region (for a large number of α to make the method effective). We should come to the conclusion telling us about the higher effectivity of the last method discussed.
We have aimed to find the optimal solution for a classical objective function in the fuzzified feasible region by studying the methods of inexact linear programming. To achieve this goal we used two different methods, which can be named as interval inexact linear programming and fuzzy number inexact linear programming. We now can conclude that fuzzy linear programming is better than the classical linear programming, because violation of a single constraint in classical programming is not acceptable. This can give the infeasible solution. But in inexact linear programming it is acceptable to have the small violations introduced in coefficients of a constraint. By using interval forms, it is possible to have e.g., a comparison and an evaluation of different costs at different shops. We conclude that the presence of a larger number of $\alpha$-cuts will be more effective for the $\alpha$-cut method adapted for the feasible region. This conclusion is motivated by the fact that the appearance of a great many $\alpha$-cuts provides the program with more thorough solutions. On the other hand the presence of too many restrictions can have a noisy character for the program algorithm. Therefore, another conclusion is drawn, namely, the L-R method will be more effective for optimization problems because L-R method reduces the number of restrictions with real numbers in feasible region.
Bibliography


