Convertible Bonds: a Qualitative and Numerical Analysis

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ABSTRACT

A convertible bond is a financial instrument which has both an equity part and a fixed-income part. The pricing of financial securities has for quite obvious reasons become extensively studied in the past decades. In this paper we study the Black-Scholes model, based on the equity value, where the equity is modelled by geometric brownian motion. We introduce the pricing of financial securities in general by Partial Differential Equation (PDE) approach. We continue by studying the convertible bond with a call feature, which is a derivative of the stock price. Our model leads to a free boundary problem together with a parabolic partial differential equation. We also give some analytical results on uniqueness and monotonicity of the solutions. This paper ends with a numerical study of the solutions for different bond features.
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1. Introduction

1.1. Background.
The problem of pricing financial securities has been extensively studied during the past decades. In this thesis we will treat our financial problems from a mathematical point of view with a PDE approach, and we aim to study the pricing of American convertible bonds a little closer. We start by explaining the pricing of financial securities in general, where the value of our security depends on the price $S$ of the underlying asset, which can be a stock, an index or a commodity. We assume that the price of this underlying asset follows a stochastic differential equation of lognormal random type

$$dS = \mu S dt + \sigma S dX.$$ (1)

The parameters $\mu, \sigma$ are given, and $X$ is the standard brownian motion. We then apply our mathematics to the case of convertible bonds, of which we do a qualitative and numerical study. For the reader who is not familiar with financial terms, we refer to Appendix A.

1.2. Historic remarks.
Convertible bonds have been used for raising capital since the mid-nineteenth century, mainly by U.S railroad companies to finance their growth. The convertible bond gave the attractive possibility to benefit from rising stock prices as well as the security in the interests and the repayment of the loan if the stock prices didn’t rise. Today, they are used at a much larger scale to a very diversified extent. In 2011 it had a volume of over 500 billion USD (BIS, Quarterly Review, Statistical Annex, December 2011), with a market dominated by U.S issuers (43%), followed by European issuer (29%), with a quick rise in activity from Asian issuers.

1.3. Convertible bonds.
A bond is a debt to an issuing company or government. It’s a debt investment in which an investor loans money to an entity. The funds are borrowed for a defined period, the issuer guarantees to pay back the investor on a specific date called the maturity date. The funds are loaned at a fixed interest rate (called coupon, and is often payed annual or semiannual). The main reason for issuing bonds is to raise capital.
The bond may have a certain feature that gives the owner the right to convert the bond to a pre-determined number of equities, and is then called a Convertible bond. These are the type of bonds we will study. There are different styles of convertible bonds, and we list 3 of them below:
- **European convertible bond** is an option that may only be exercised at the maturity date, the bondholder has the right to exchange the bond into the given number of shares at this date.
- **American convertible bond** is a bond that may be exercised at any time before maturity.
- **Bermudan convertible bond** is a bond that may be exercised at discrete times before or at maturity.

We are mainly interested in the American Convertible Bond. The bond may also have a call option which gives the right to the issuer to call back the bond for a certain amount. Callable bonds are less worth than regular bonds as they give an extra right to the issuer. In this paper we aim to present a mathematical model for pricing convertible bonds with a call feature and we will try to establish properties of the solutions such as uniqueness and monotonicity. We also give some numerical results.

## 2. Pricing financial securities

Through this paper we will use the following nomenclature:

### Nomenclature

<table>
<thead>
<tr>
<th>Variable</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D$</td>
<td>The dividend rate (related to the stock)</td>
</tr>
<tr>
<td>$K$</td>
<td>The coupon rate (related to the bond)</td>
</tr>
<tr>
<td>$S$</td>
<td>The asset price</td>
</tr>
<tr>
<td>$t$</td>
<td>The time</td>
</tr>
<tr>
<td>$T$</td>
<td>The maturity date</td>
</tr>
<tr>
<td>$V(S,t)$</td>
<td>The price of a bond</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>The volatility of S (the standard deviation of the return of the asset)</td>
</tr>
<tr>
<td>$\mu$</td>
<td>The drift rate of S, the average rate of growth of the asset price</td>
</tr>
<tr>
<td>$\Pi$</td>
<td>The value of a portfolio</td>
</tr>
<tr>
<td>$r$</td>
<td>The interest rate</td>
</tr>
<tr>
<td>$dX$</td>
<td>A random variable, known as a Wiener process</td>
</tr>
<tr>
<td>$\mu$</td>
<td>The drift rate of S, the average rate of growth of the asset price</td>
</tr>
<tr>
<td>$Z_f$</td>
<td>The face value of the bond (amount payed to the holder at maturity time)</td>
</tr>
<tr>
<td>$Z_c$</td>
<td>The call value of the bond (amount for which the company buys back the bond)</td>
</tr>
<tr>
<td>$n$</td>
<td>The conversion factor</td>
</tr>
<tr>
<td>$D_T$</td>
<td>$D_T = (0,T) \times (0,Z/n)$</td>
</tr>
</tbody>
</table>

Our assumption during all our derivations are the following:
S follows the lognormal random walk (1),
No transactions costs,
No arbitrage: all risk-free portfolios must earn the same return,
Shortselling is permitted and assets are divisible,
The existence of convertible bonds does not affect the market worth of a company.

The basic question is how we can predict the future price of an asset. Let us denote the asset price at time $t$ by $S$. During the time period $dt$, $S$ will change to $S + dS$. There is one predictable part of the future price which comes from the return of money in a risk-free bank:

$$\mu dt,$$

where $\mu$ is called the drift, a measurement of the average growth of the asset price. In simple models $\mu$ may be seen as a constant, but in more complicated models it can be a function of $S$ and $t$.

The second unpredictable part of the future asset price comes from the random changes in the price and is represented by a random variable drawn from a normal distribution with mean zero:

$$\sigma dX,$$

where $\sigma$ is called the volatility, a measurement of the standard deviation of the returns. This together gives us an expression for the change $dS$ in the asset price:

$$dS = \sigma dX + \mu dt.$$

The variable $dX$ contains the randomness of the asset price. We state the very useful result on $dX$:

$$dX^2 \to dt \quad as \quad dt \to 0.$$  

We will be needing another tool through our derivations: Itô’s lemma, see [WHD96]. This lemma relates a small change in a function of a random variable to the change of the variable itself. If $f$ is a function of a random variable $G$ that follows a stochastic differential equation of the form

$$dG = A(G, t)dX + B(G, t)dt$$
then for a given \( f(G) \) we have:

\[
d f = A \frac{df}{dG} dX + \left( B \frac{df}{dG} + \frac{1}{2} A^2 \frac{d^2 f}{dG^2} \right) dt.
\]

For the case where our function \( f \) depends on the random variable \( S \) which follows the stochastic differential equation (1), we get:

\[
d f = \sigma S \frac{df}{dS} dX + \left( \mu S \frac{df}{dS} + \frac{1}{2} \sigma^2 S^2 \frac{d^2 f}{dS^2} \right) dt.
\]

The expression for \( df \) can be generalised to a function of two variables, the random variable \( S \) and the time, \( t \), i.e. \( f(S, t) \). The Taylor expansion of \( f(S + dS, t + dt) \) now becomes

\[
d f = \frac{\partial f}{\partial S} dS + \frac{\partial f}{\partial t} dt + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} dS^2.
\]

If we now use expressions (1) and (3) the equation becomes

\[
d f = \sigma S \frac{\partial f}{\partial S} dX + \left( \mu S \frac{\partial f}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + \frac{\partial f}{\partial t} \right) dt.
\]

This is the Taylor expansion of functions of random variables and we will use it when we derive our equations describing the value of our securities. We describe shortly the European and American option before turning our focus on the convertible bond.

An important concept in the pricing of financial securities is the concept of arbitrage: *when the future price of an investment asset is unknown, we assume that its future market price is determined by the price of another asset whose future price is deterministic, for example the amount of money one can get from putting it in a risk free bank account with a known interest rate \( r \).* Shortly put, one cannot make money without risk.

### 2.1. The European option.

In this section we describe the pricing of a European option. A European option is a privilege, a contract, which gives the buyer the right, but not the obligation, to buy (for a call option) or to sell (for a put option) the underlying asset for a certain price, which is predetermined. The underlying can be shares, currencies, rates, commodities, etc. Arbitrage reasoning together with Ito’s lemma leads us to the Black Scholes equation for the European option, with boundary condition. We do not include the derivation in the paper as we perform a similar derivation for convertible bonds. The European problem:
\[ \frac{\partial C}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS \frac{\partial C}{\partial S} - rC = 0 \]

\[ C(0, t) = 0 \]
\[ C(S, t) \sim S \quad \text{as} \quad S \to \infty \]
\[ C(S, T) = \max (S - E, 0). \]

The solution can be derived using a change of variable, see Appendix B. If we assume the interest rate, \( r \), and the volatility, \( \sigma \) constant, then we have the exact solution for the European call:

\[ C(S, t) = SN(d_1) - Ee^{-r(T-t)}N(d_2), \]

where \( N(\cdot) \) is the cumulative distribution function for a standardised normal random variable, given by

\[ N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}y^2} dy. \]

We have \( d_1 \) and \( d_2 \) as follows

\[ d_1 = \frac{\log (S/E) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma \sqrt{T-t}} \]
\[ d_2 = \frac{\log (S/E) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma \sqrt{T-t}}. \]

For a put we have the solution

\[ P(S, t) = Ee^{-r(T-t)}N(-d_2) - SN(-d_1). \]

2.2. The American option.

We now introduce the American option, which has the early exercise option as opposed to the European option. American options can be modelled as an obstacle problem with the following set of constraints:

- \( V(S, t) \geq \max (S - E, 0) \), the B-S equation is replaced by an inequality.
- \( \frac{\partial V}{\partial S} \) must be continuous.
They follow from arbitrage and the last one is given by an informal financially based argument in [WHD96]. We can write the American problem as a free boundary problem by dividing our region of S in two: one where early exercise is optimal and one where early exercise is not. Our obstacle is the payoff function
\[ g(S, t) = \max(S - E, 0) \] (for a call) or
\[ g(S, t) = \max(E - S, 0) \] (for a put) and our region is divided in two by the optimal exercise price \( S_f(t) \), which we do not know the position of.

The American option with dividend:

- for the interval where exercise is optimal, \( V(S, t) = g(S, t) \) and
  \[ \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (rS - D(S, t)) \frac{\partial V}{\partial S} - rV < 0. \]
- for the interval where exercise is not optimal, \( V(S, t) > g(S, t) \) and
  \[ \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (rS - D(S, t)) \frac{\partial V}{\partial S} - rV = 0. \]
- at the free boundary \( S_f(t) \), \( V(S_f(t), t) = g(S_f(t)) \) and \( \frac{\partial V}{\partial S}(S_f(t), t) = -1. \)

The value of an American Option is much more complicated to determine than for an European Option, since it has free boundary conditions. Even so it can be done by linear complementary, where we can eliminate the explicit mention of the free boundary points. Let us use the transformation of variables from \((S, t)\) to \((x, \tau)\):

\[
S = E e^x, \quad t = T - \tau/\frac{1}{2} \sigma^2.
\]

The optimal exercise price \( S = S_f(t) \) is then represented by \( x = x_f(\tau) \), where \( S_f(t) < E \) and \( x_f(\tau) < 0 \). The payoff function \( \max(E - S, 0) \) becomes, by setting \( k = \frac{1}{\frac{1}{2} \sigma^2} \):

\[
g(x, \tau) = e^{\frac{1}{2} (k+1)^2 \tau} \max \left( e^{\frac{1}{2} (k-1)x} - e^{\frac{1}{2} (k+1)x}, 0 \right).
\]

Since the \( x \) axis is divided into two distinct regions, one where the Black-Scholes equation holds and one where the option value equals the payoff function, we obtain the following equation

\[
\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} \quad \text{for} \quad x > x_f(\tau),
\]
\[
u(x, \tau) = g(x, \tau) \quad \text{for} \quad x \leq x_f(\tau).
\]

The initial condition and the asymptotic behavior follows
\[ u(x, 0) = g(x, 0) = \max \left( e^{\frac{1}{2}(k-1)x} - e^{\frac{1}{2}(k+1)x}, 0 \right), \]

\[ \lim_{x \to \infty} u(x, \tau) = 0, \]
\[ \lim_{x \to -\infty} u(x, \tau) = g(x, \tau). \]

We also have the fact that \( u \) and \( \partial u / \partial x \) must be continuous and the constraint

\[ u(x, \tau) \geq g(x, \tau). \]

We use the restriction \(-x^- < x < x^+\), where \( x^- \) and \( x^+ \) are large. This gives the boundary conditions

\[ u(x^+, \tau) = 0, \]
\[ u(x^-, \tau) = g(x^+, \tau). \]

This means that we replace the exact boundary conditions by the approximation that for values of \( S \) near zero, \( P \) becomes the payoff function (i.e. \( E - S \)), while for large values of \( S \), \( P \) equals zero.

In order to write this as a linear complementary problem, we establish that our obstacle is the payoff function \( g(x, \tau) \). This means that \( u(x, \tau) \) will either equal \( g(x, \tau) \) or the Black-Scholes equation will hold. Since the B-S equation becomes the diffusion equation in the variables \( u(x, \tau) \), the following holds

\[ \left( \frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2} \right) \cdot (u(x, \tau) - g(x, \tau)) = 0, \]
\[ \left( \frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2} \right) \geq 0, \quad (u(x, \tau) - g(x, \tau)) \geq 0. \]

With initial and boundary conditions

\[ u(x, 0) = g(x, 0) = \max \left( e^{\frac{1}{2}(k-1)x} - (e^{\frac{1}{2}(k+1)x}), 0 \right), \]
\[ u(-x^-, \tau) = g(-x^-, \tau), \]
\[ u(x^+, \tau) = g(x^+, \tau) = 0. \]

We also have the condition that \( u(x, \tau) \) and \( \partial u / \partial x \) are continuous.

This is the linear complementary form of an American Put Option. The two possibilities, \( u(x, \tau) - g(x, \tau) = 0 \) and \( (\partial u / \partial \tau - \partial^2 u / \partial x^2) = 0 \), corresponds
to cases where it is optional to exercise the option \( u = g \) respectively where it is not \( u > g \). The reason why this formulation of the problem is so advantageous is the fact that the free boundary does not need to be tracked explicitly.

3. Convertible bonds

3.1. The bond pricing equation: PDE approach.

3.1.1. Basics.

In the first sections we discussed the pricing of financial securities in general and we gave the example of the European and American option. We now introduce the bond. A bond is a type of loan, a debt investment in which an investor loans money to an entity (corporate, government). It is a fixed income security, at the maturity date one will get the loaned funds back with a cash dividend. One can also get coupon payment through the lifetime of the bond. The maturity date and the coupon are known. As bonds usually have a longer life time than securities such as options, it is not as plausible to assume constant or known interest rates. One can choose to model the interest rate as a random variable and derive a partial differential equation that models the value of the bond, but this gives a complicated situation for analytical results, we therefore only consider known interest rates in our study.

We denote \( V = V(t) \) the value of the bond: it is a function of time only because the interest rate \( r(t) \) and the coupon payment \( K(t) \) are functions of time (we ignore the maturity date dependence)

\[
\left( \frac{dV(t)}{dt} + K(t) \right) dt.
\]

This describes the change of value of the bond during a time \( dt \) plus the received coupon payment during that interval, giving us the total change in holding. The arbitrage principle tells us to set this amount equal to the returns from a bank:

\[
\frac{dV(t)}{dt} + K(t) = r(t)V.
\]

This is easily solved with the integrating factor.

\[
V(t) = e^{\int_t^T r(\tau) d\tau} \left( V(T) + \int_t^T K(t') e^{\int_t^{t'} r(\tau) d\tau} dt' \right).
\]
For zero-coupon bonds, \( K(t) = 0 \) and
\[
V(t) = V(T)e^{\int_t^T \tau(\sigma) \, d\sigma}.
\]
For discretely paid coupons, \( K(t) \) can be written as a sum of delta functions. Consider the case of one payment \( K_c \) at time \( t_c \leq T \), we can substitute the delta function in (5):
\[
V(t) = e^{\int_t^T \tau(\sigma) \, d\sigma} \left( V(T) + K_c H(t_c - t) \int_t^{t_c} K(t')e^{\int_t^{t'} \tau(\sigma) \, d\sigma} \, dt' \right).
\]
This gives the value of the discrete coupon payment bond. We continue our study on bonds by considering the convertible bond.

3.1.2. The convertible bond pricing equation.
A convertible bond is a bond that has the additional feature that the bond may be converted by the owner to a pre-determined number of a specific asset at any time. The convertible bond on an underlying asset with price \( S \) returns \( Z \) at the maturity date \( T \) unless the bond has been converted into \( n \) of the underlying asset before the maturity date. Now the bond price depends on the value of the underlying asset \( S \), so \( V = V(S, t) \): Considering a portfolio that is one bond long and \( \Delta \) assets short, \( \Pi = V - \Delta S \).
\[
d\Pi = dV - \Delta dS
\]
is the change in the value of the portfolio for a time step \( dt \). If the underlying asset pays a divident \( D(S, t) \) and a coupon \( K(S, t) \) we have:
\[
d\Pi = dV - \Delta(dS + Ddt) + K(S, t)dt.
\]
Itô's lemma gives us the change in \( V(S, t) \) as \( S \) follows the random walk \( dS = \sigma S dX + \mu S dt \) or \( dS = \sigma S dX + \mu S dt - D(S, t)dt \) (the asset price must fall by the amount of the dividend payment):
\[
d\Pi = \sigma S \frac{\partial V}{\partial S} dX + \left( \frac{\partial V}{\partial t} + (\mu S - D(S, t)) \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt
- \Delta(dS + D(S, t)dt) + K(S, t)dt
= \sigma S \left( \frac{\partial V}{\partial S} - \Delta \right) dX + \frac{\partial V}{\partial t} dt + \mu S \left( \frac{\partial V}{\partial S} - \Delta \right) dt
+ \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} dt
- D(S, t) \frac{\partial V}{\partial S} + K(S, t)dt.
\]
We then choose \( \Delta = \frac{\partial V}{\partial S} \) to eliminate the random component \( dX \):
We use arbitrage arguments to set
\[ d\Pi = \frac{\partial V}{\partial t} dt + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} dt - D(S, t) \frac{\partial V}{\partial S} + K(S, t). \]

as the change in holdings should be equal or less than the return from a risk free bank. This gives us the convertible bond pricing equation:

\[ d\Pi \leq r\Pi dt \]
\[ \leq r(V - \Delta S) dt, \]

We have the terminal condition \( V(S, T) = \max(nS, Z_f) \) and the constraint \( V \geq nS \). The behavior at infinity should be:
\[ \lim_{S \to \infty} V(S, t) = nS \]
if the stock price goes up, one would want to convert rather than receive \( Z \). On the other hand, if \( S = 0 \) (no underlying asset) then the equation becomes
\[ \frac{\partial V}{\partial t} + K = rV \]
and the solution to this ODE is
\[ V(t) = e^{\int_t^T r(\tau) d\tau} \left( Z + \int_t^T K(t') e^{\int_t^{t'} r(\tau) d\tau} dt' \right) = Ze^{-r(T-t)} + K/r - K/re^{-r(T-t)}. \]

One should note that it is not obvious that the solution goes to this when we let \( S \) go to 0 because we would need functions satisfying
\[ \lim_{S \to 0} S^2 V_{SS} = \lim_{S \to 0} SV_S = 0. \]

We consider only solutions that satisfy this.

So we have a free boundary problem. We can reformulate our problem by considering different regions: the continuation region \( \{(S, t) : nS < V(S, t) < \infty\} \) (where the bond should be held) and the conversion region \( \{(S, t) : S \geq 0, V = nS\} \) (where the bond should be converted to the shares). From now on we assume a constant dividend yield such that
\[ D(S,t) = DS. \]

Define the operator \( \mathcal{L} = -\frac{\partial}{\partial t} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} - (r - D)S \frac{\partial}{\partial S} + r. \)

Our variational equality problem is the following

\[
\begin{cases}
\mathcal{L}V = K, & \{nS < V < \infty\} \\
\mathcal{L}V \geq K, & \{V = nS\} \\
V(S,T) = \max(nS,Z) & 0 \leq S \leq \infty
\end{cases}
\]

where we treat the boundary value at \( S = 0 \) as above using assumption (8) and the value can be computed.

### 3.2. Call feature.

A bond with a call feature, a callable bond, gives the right to the issuer to call back the bond for a certain amount, the call value of the bond. Here, we will take \( Z_f = Z_c = Z \), the face value equals the call value. We have the extra constraint \( V \leq Z \) because the issuer does not want the holder to have a security worth \( V > Z \) that the issuer can call back for \( Z \). This happens when \( S \geq Z/n \) and because \( V \geq nS \) then \( V \geq Z \) and as discussed this is not desirable. We then restrain our region to \( S \leq Z/n \), and we define \( D_T = (0,T) \times (0,Z/n) \).

Our variational inequality formulation becomes:

\[
\begin{cases}
\mathcal{L}V = K, & \{nS < V < Z\} \cap D_T \\
\mathcal{L}V \geq K, & \{V = nS\} \cap D_T \\
V(Z/n,t) = Z, & 0 \leq t \leq T \\
V(S,T) = Z, & 0 \leq S \leq Z/n
\end{cases}
\]

Figure 1 illustrates the different regions:

### 4. Qualitative analysis

#### 4.1. Preliminaries.

##### 4.1.1. Definitions and properties.

In order to be able to give a proper mathematical analysis of our problem we define a couple of concepts and state some properties in the context of parabolic PDE. We refer to [Frie82] for theory on variational inequalities and free-boundary problems.
Definition 1. (WEAK DERIVATIVE) Let \( u \) be an absolutely integrable function on \( \Omega \), a bounded domain of \( \mathbb{R}^n \). For \( \alpha = (\alpha_1, \ldots, \alpha_n) \), \( v \) is the \( \alpha \)-th derivative of \( u \), \( D^\alpha u \), if

\[
\int_\Omega u D^\alpha \phi \, d\mu = (-1)^{|\alpha|} \int_\Omega v \phi \, d\mu
\]

for all infinitely differentiable functions \( \phi \) with compact support in \( \Omega \).

Definition 2. (SOBOLEV SPACE) The Sobolev space \( W^{k,p}(\Omega) \) is defined as the set of functions such that the functions themselves and their weak derivatives up to order \( k \) lie in \( L^p(\Omega) \), i.e

\[
W^{k,p}(\Omega) = \{ u \in L^p(\Omega) | D^\alpha u \in L^p(\Omega) \forall |\alpha| \leq k \}.
\]

Definition 3. (SUPER/SUB-SOLUTION) We say that \( u \) is a supersolution (resp. subsolution) to a parabolic equation described by the operator \( \mathcal{L} \) with Dirichlet boundary conditions at a point \( x_0, t \) if it is lower semi-continuous (resp. upper semi-continuous) and for all testfunctions \( \phi \) such that \( u - \phi \) has a local minimum (resp. maximum) in \( x_0 \), we have

\[ \mathcal{L}(\phi) \geq 0 \text{ (resp. } \mathcal{L}(\phi) \leq 0 \text{)} . \]

For a more geometric explanation we can see it as \( \phi \) touches the graph of \( u \) at \( x_0, t \) from below resp. above. We essentially require the derivatives of \( \phi \) to fulfill the corresponding inequalities.
Theorem 1. *(STRONG MAXIMUM/MINIMUM PRINCIPLE)*
Let $\mathcal{L}$ be a parabolic operator and let $\mathcal{L}u \leq 0$ (u is a subsolution to a parabolic equation) in the bounded domain $\Omega$. Then $u$ cannot reach a maximum value in the interior of $\Omega$ unless $u$ is constant. In other words, $u$ reaches a maximum on the boundary only.

Let $\mathcal{L}$ be a parabolic operator and let $\mathcal{L}u \geq 0$ (u is a supersolution to a parabolic equation) in the bounded domain $\Omega$. Then $u$ cannot reach a minimum value in the interior of $\Omega$ unless $u$ is constant. In other words, $u$ reaches a minimum on the boundary only.

Theorem 2. *(COMPARISON PRINCIPLE)*
Let $\mathcal{L}$ be a parabolic operator, let $u$ be a supersolution to the parabolic equation $\mathcal{L}u = 0$ and let $v$ be a subsolution to the same equation, ie $\mathcal{L}u \geq 0$ and $\mathcal{L}v \leq 0$.
If $v \leq u$ on the boundary $\partial \Omega$ then the inequality holds in $\Omega$.

Theorem 3. *(SARD’S THEOREM)*
The critical values (the image of the set of critical points) of a smooth function from one Euclidean space or manifold to another has Lebesgue measure zero, i.e. they form a null set.

With these tools we are now able to give the proof to some results about the solutions.

4.1.2. *Restatement of the problem.*
To be able to use general theory of parabolic PDE, we want to transform the backward equation into a forward parabolic PDE. This is done by the change of variables which is suggested on page 11 in Paper 1 in [Saj13].

\[ \tilde{V}(S, t) = V(-S + Z/n, T - t). \]

This leads to the following equation

\[
\begin{cases}
\tilde{L}\tilde{V} = K, & D_T \cap \{-nS + Z < V\} \\
\tilde{L}\tilde{V} \geq K, & D_T \\
\tilde{V}(0, t) = Z, & 0 \leq t \leq T \\
\tilde{V}(S, 0) = Z & 0 \leq S \leq Z/n
\end{cases}
\]

where \( \tilde{L} \) is the following operator

\[
\tilde{L} = \partial_t - \frac{1}{2} \sigma^2 (-S + Z/n)^2 \partial_{SS} + (r - D)(-S + Z/n) \partial_S + r.
\]

4.2. Existence and uniqueness of \( V \).

**Proposition 1.** There exists a solution \( V \in W^{2,p}_x \cap W^{1,p}_t \), \( p \in ]1; \infty[ \) to Problem 9. This solution is unique when considering solutions satisfying (8).

For the proof of the existence of the solution, one can use the classical penalty method as described in [YiYang11].

**Uniqueness:** This proof is adapted from Paper I in [Saj13]. Assume there exists two solution \( \tilde{V}_1 \) and \( \tilde{V}_2 \) to the problem 11. Define the set

\[ \Omega = \{(S, t) : \tilde{V}_1(S, t) > \tilde{V}_2(S, t)\} \subset D_T. \]

We have the constraint \( \tilde{V}_2 \geq -nS + Z \) and \( \tilde{V}_1 > \tilde{V}_2 \) in \( \Omega \) and therefore \( \tilde{L}\tilde{V}_1 = K \) in \( \Omega \). We also have \( \tilde{L}\tilde{V}_2 \geq K \) in \( \Omega \), ie \( \tilde{V}_2 \) is a super-solution to the equation. The boundary of \( \Omega \) can either lie in the interior of \( D_T \) or have a part of its boundary in common with \( D_T \). This is illustrated in Figure 3.
Since $V_1 > V_2$ in $\Omega$ and we assume $V_1$ and $V_2$ continuous, we have that $V_1 = V_2$ on the boundary of $\Omega$ if the boundary of $\Omega$ lies in the interior of $D_T$. Because of the possible case when $\Omega$ has a part of its boundary in common with $D_T$, we have to study the values of $V_1$ and $V_2$ on the boundary of $D_T$. Since $\tilde{V}_1$ and $\tilde{V}_2$ satisfy the boundary and initial condition in (11) we have that

$$
\tilde{V}_1(0, t) = \tilde{V}_2(0, t),
\tilde{V}_1(S, 0) = \tilde{V}_2(S, 0).
$$

Now we must consider the last part of the boundary i.e. $S = Z/n$. The equation $\tilde{L}\tilde{V} = K$ becomes

$$
\frac{\partial \tilde{V}}{\partial t} + r\tilde{V} = K.
$$

This is an ODE with solution

$$
\hat{V}(Z/n, t) = Ze^{-rt} + \frac{K}{r}(1 - e^{-rt}), \quad 0 \leq t \leq T,
$$

and thus

$$
\hat{V}_1(Z/n, t) = \hat{V}_2(Z/n, t).
$$

That means $\hat{V}_1 = \hat{V}_2$ on the whole boundary of $\Omega$ and because $\hat{V}_2$ is a supersolution while $\hat{V}_1$ is a solution, the comparison principle gives us that

---

**Figure 3.** The case when $\Omega$ lies in the interior of $D_T$ is illustrated to the left and the case when $\Omega$ have a part of its boundary in common with $D_T$ is illustrated to the right.
\( \tilde{V}_1(S,t) \leq \tilde{V}_2(S,t) \) in \( \Omega \). This is a contradiction, and therefore we have a unique solution.

4.3. Monotonicity of \( V \), in \( S,t \).

**Proposition 2.** The solution \( V \in W^{2,p}_x \cap W^{1,p}_t \), \( p \in ]1;\infty[ \) to Problem 9 is monotone in both \( t \) and \( S \) direction.

We start with proving the monotonicity in the \( t \) direction by considering

\[
\tilde{V}_\alpha(S,t) = \tilde{V}(S,t + \alpha).
\]

This will not change the operator, \( \tilde{L} \), since its coefficients do not depend on \( t \). We want to show that \( \tilde{V}_\alpha \leq \tilde{V} \) in \( D_{T-\alpha} \) for \( \alpha > 0 \). Similarly as above we consider the boundaries. From equation (11) it is clear that

\[
\tilde{V}_\alpha(0,t) = \tilde{V}(0,t).
\]

Further we have that \( \tilde{V}(S,0) = Z \) and due to callability (see Section 3.2) \( Z \) is the maximum value of \( \tilde{V} \). This means that

\[
\tilde{V}_\alpha(S,0) \leq \tilde{V}(S,0).
\]

On the boundary \( S = Z/n \), expression (12) gives us

\[
\tilde{V}(Z/n,t) = \frac{K}{r} + \left( Z - \frac{K}{r} \right) e^{-rt},
\]

\[
\tilde{V}_\alpha(Z/n,t) = \frac{K}{r} + \left( Z - \frac{K}{r} \right) e^{-r(t+\alpha)}.
\]

Since we have the restriction \( K < rZ \), it follows that \( Z - K/r > 0 \) and thus we have that

\[
\frac{K}{r} + \left( Z - \frac{K}{r} \right) e^{-r(t+\alpha)} < \frac{K}{r} + \left( Z - \frac{K}{r} \right) e^{-rt}
\]

This means that

\[
\tilde{V}_\alpha(Z/n,t) < \tilde{V}(Z/n,t).
\]

We have concluded that \( \tilde{V}_\alpha \leq \tilde{V} \) on the boundary and hence the comparison principle gives us that \( \tilde{V}_\alpha \leq \tilde{V} \) on \( D_{T-\alpha} \). This means that for the original problem we have that
and the proof is done.

Further we want to prove monotonicity in the $S$ direction. A displacement similar to the one in the $t$ direction would change the operator, so we must use another approach. Consider the problem (9). We have no boundaries at $S = 0$ and the behavior of the solution in that point is rather difficult to determine since the operator have singularities there. A way to treat this is to expand the definition domain of the equation to $\{-Z/n < S < Z/n, 0 < t < T\}$ as suggested in [Saj13] by considering the evenly reflected coefficients of the operator in $\{-Z/n < S < 0, 0 < t < T\}$. Further we can rewrite the operator as uniformly parabolic (i.e. the coefficient in front of $\partial_{SS}$ is not zero) in the following way

$$L_\epsilon = -\partial_t - \frac{1}{2} \sigma^2 (S^2 + \epsilon) \partial_{SS} - (r - D) S \partial_S + r,$$

where $\epsilon > 0$. If we can prove nice properties for this operator and then let $\epsilon \to 0$, we might be able to prove the same properties for the limit operator. The problem (9) with the $L_\epsilon$ operator will have an unique solution which we denote $W_\epsilon$. By symmetry in $S$ it follows that

$$\partial_S W_\epsilon(0, t) = 0.$$  \hspace{1cm} (14)

Now we use the same substitution of variables as above

$$\tilde{W}_\epsilon = W_\epsilon(-S + Z/n, T - t) \quad \text{in } D_T.$$  \hspace{1cm} (15)

(14) becomes

$$\partial_S \tilde{W}_\epsilon(Z/n, t) = 0.$$  \hspace{1cm} (15)

Since $\tilde{W}_\epsilon$ is a solution of (11), it holds that $\partial_S \tilde{L}_\epsilon(\tilde{W}_\epsilon) = \partial_S c$ in the region $\{\tilde{W}_\epsilon > -Sn + Z\}$.

We have that:
\[ \partial_S \mathcal{L}_\epsilon(\tilde{W}_\epsilon) = \partial_S (\partial_t \tilde{W}_\epsilon) - \frac{1}{2} \sigma^2 \partial_S \left( ((-S + Z/n)^2 + \epsilon) \partial_{SS} \tilde{W}_\epsilon \right) + \\
+ (r - D) \partial_S \left( (-S + Z/n) \partial_S \tilde{W}_\epsilon \right) + \partial_S (r \tilde{W}_\epsilon) \\
= \partial_t (\partial_S \tilde{W}_\epsilon) - \frac{1}{2} \sigma^2 \left( ((-S + Z/n)^2 + \epsilon) \partial_{SS} (\partial_S \tilde{W}_\epsilon) - 2(-S + Z/n) \partial_S \tilde{W}_\epsilon \right) + \\
+ r \partial_S \tilde{W}_\epsilon \\
= \partial_t (\partial_S \tilde{W}_\epsilon) - \frac{1}{2} \sigma^2 ((-S + Z/n)^2 + \epsilon) \partial_{SS} (\partial_S \tilde{W}_\epsilon) + \\
+ (\sigma^2 + r - D)(-S + Z/n) \partial_S (\partial_S \tilde{W}_\epsilon) + D \partial_S \tilde{W}_\epsilon = 0. \]

This means that \( \tilde{W}_\epsilon \) satisfies

\[
\partial_t (\partial_S \tilde{W}_\epsilon) - \frac{1}{2} \sigma^2 ((-S + Z/n)^2 + \epsilon) \partial_{SS} (\partial_S \tilde{W}_\epsilon) + \\
(\sigma^2 + r - D)(-S + Z/n) \partial_S (\partial_S \tilde{W}_\epsilon) + D \partial_S \tilde{W}_\epsilon = 0
\]

in the region \( \{ W_\epsilon > -Sn + Z \} \). This means that \( \partial_S \tilde{W}_\epsilon \) is a solution to a parabolic differential equation. Further for \( S = Z/n \) it follows from (15) that \( \partial_S \tilde{W}_\epsilon = 0 \). For \( \partial \{ \tilde{W}_\epsilon > -Sn + Z \} \) and for \( t = 0 \) we have that \( \partial_S \tilde{W}_\epsilon = -n \). Finally on \( S = 0 \) for \( 0 < t < T - t_0 \), since \( \tilde{W}_\epsilon \geq -Sn + Z \), we have that \( -n \leq \partial_S \tilde{W}_\epsilon \leq 0 \). Those values of \( \partial_S \tilde{W}_\epsilon \) on the boundaries are illustrated in figure 4.

**Figure 4.** Illustration of the values of \( \partial_S \tilde{W}_\epsilon \) on the boundaries.
If we now apply the maximum and minimum principle we can conclude that
\[-n \leq \partial_S \bar{W}_\epsilon \leq 0.\]
This shows that $\bar{W}_\epsilon$ is decreasing with increasing $S$ in the continuation region, which means that $W_\epsilon$ is increasing with increasing $S$. As $\epsilon$ tends to zero, $W_\epsilon$ tends to $V$ and we obtain the same properties for $V$, i.e. $V$ is monotone (increasing) in the $S$-direction in the continuation region. In the conversion region on the other hand we have that $V = nS$, i.e. $\partial_S V = n$. This means that $V$ is monotone increasing in the $S$, but grows slower (or equal growth rate) than $nS$, i.e.
\[0 \leq V_S \leq n,\]
and the proof is done.

4.4. Bond Convexity.

We studied the convexity of the optimal stopping boundary $S(t)$ inspired by [E.Ekstrom04] which proves the convexity of the continuation region for an American Put Option. This turned out to be harder than expected to apply to convertible bonds. We used a transformation of variables and tried to prove some properties for the transformed problem. We have not yet managed to show that this implies convexity of the original problem, but we think it should be possible to do so. The rest of the section studies the transformed problem.
Consider the optimal stopping boundary $S(t)$ which delimits the continuation region from the conversion region. We want to prove that $S(t)$ is a convex function. Assume $S(t)$ smooth. First recall equation (9), in the continuation region, it can be formulated as follows
\[
\begin{align*}
\mathcal{L}V &= K, & S < S(t) \\
V(S, t) &= nS, & S = S(t) \\
V_s(S, t) &= n, & S = S(t) \\
V(S, T) &= Z & 0 \leq S \leq Z/n
\end{align*}
\]
(16)
Let us do a substitution of variables so we can work with $f(x, \tau)$ and $x(\tau)$ instead of $V(S, t)$ and $S(t)$:
\[
\begin{align*}
V(S, t) &= nS + Zf(x, \tau) \\
S(t) &= Ze^{x(\tau)}
\end{align*}
\]
where
\[
\begin{align*}
S &= Ze^x \\
\tau &= T - \frac{2\pi}{\sigma^2}
\end{align*}
\]
\[\iff x = \ln \frac{S}{Z}, \quad \tau = \frac{\sigma^2}{2} (T - t).\]

Let us also define \(\Psi = \{x : x = g(\tau)\}\). On the free boundary we have that \(V = nS\), so

\[(17) \quad f(\tau, x(\tau)) = 0.\]

Since \(V\) is continuous, so is \(f\) and we can conclude that

\[
\begin{align*}
f &> 0 \quad \text{for} \quad x < x(\tau) \\
f &= 0 \quad \text{for} \quad x \geq x(\tau)
\end{align*}
\]

and equation (16) now becomes

\[
\left\{
\begin{array}{ll}
\hat{L}f = K, & x < x(\tau) \\
f(\tau, x) = 0, & x = x(\tau) \\
f_x(\tau, x) = 0, & x = x(\tau) \\
f(x, 0) = 1 - \frac{ne^{x}}{Z}, & -\infty < x \leq -\ln n
\end{array}
\right.
\]

\[(18)\]

where
\[\hat{L}f = Z\left(\frac{\sigma^2}{2} f_\tau - \frac{\sigma^2}{2} f_{xx} + \left(\frac{\sigma^2}{2} - r\right)f_x + rf\right).\]

We start by recalling that \(V\) is monotone increasing in the \(t\)-direction and \(S\)-direction (Proposition 2 with proof) i.e. \(V_t \geq 0, V_S \geq 0\). We can then draw some conclusions on the signs of \(f_x\) and \(f_\tau\). Differentiate the expression for \(V(S, t)\) defined above with respect to \(S\) and \(t\):

\[(19) \quad V_S(S, t) = n + \frac{1}{S} Z f_x(x, \tau),\]

\[(20) \quad V_\tau(S, t) = -\frac{\sigma^2}{2} Z f_\tau(x, \tau).\]

Because of the monotonicity of \(V\), the left-handside of (20) is non-negative, so we can draw the conclusion that \(f_\tau\) is non-positive (\(Z > 0\)).
What’s more, from the proof of monotonicity we have that $0 \leq V_S \leq n$ which yields

$$-n \leq V_S - n \leq 0.$$  

This can be compared with the terms in (19) and helps us come to the conclusion that $f_\tau$ is non-positive in the hold region. So we have that

(21) $f_\tau \leq 0$

(22) $f_x \leq 0$.

Let

(23) $R(x, \tau) = x - x(\tau)$,

we can now conclude that

(24) $f(x, \tau) = R^2(x, \tau)h(x, \tau)$

holds if $h > 0$ is a two times differentiable function, bounded near $\Psi$. We also have assumed $\Psi$ smooth, which means that $f$ is $C^2$ in $x$ and $\tau$. Let us now define

(25) $v(x, \tau) = \frac{f_\tau}{f_x}$.

Differentiating (24) and (23) gives us

(26) $f_\tau = 2RR_\tau h + R^2h_\tau$

(27) $f_x = 2RR_x h + R^2h_x$

(28) $R_\tau = -\dot{x}(\tau)$

(29) $R_x = 1$.

This inserted in (5) gives us

$$v(x, \tau) = \frac{-2R\dot{x}(\tau)h + R^2h_x}{2Rh + R^2h_x} = \frac{Rh_\tau - 2\dot{x}(\tau)h}{2h + Rh_x}.$$ 

Using the fact that $R = 0$ on $\Psi$ we obtain

$$v(x, \tau) = -\dot{x}(\tau)$$
Since (21) and (22) holds, we have that \( v(x, \tau) \geq 0 \) and thus

\[
(30) \quad \dot{x}(\tau) \leq 0.
\]

From this one should be able to find the sign of \( \dot{x}(\tau) \), perhaps by using the equation, but this is beyond the scope of this work and is an area for further research.

5. Numerical study

For a numerical approach of the problem, we used a finite difference method, the implicit Euler method. Recall the differential equation for the value of a CB:

\[
(31) \quad \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D) S \frac{\partial V}{\partial S} - rV + K \leq 0
\]

First we want to transform the equation to backward time. This is done by the change of variables \( \tau = T - t \). The equation becomes:

\[
-\frac{\partial V}{\partial \tau} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D) S \frac{\partial V}{\partial S} - rV + K = 0
\]

The time intervall is discretized as follows

\[
0 = t_0 < t_1 < \ldots < t_K = T
\]

Further we define

\[
\Delta t_k = t_{k+1} - t_k, \quad k = 0, 1, \ldots, K - 1
\]

For simplicity let

\[
\Delta t_0 = \Delta t_1 = \cdots = \Delta t_{K-1}
\]

For the backward timepoints set \( \tau_k = t_{K-k} \) for \( k = 0, 1, \ldots, K \), i.e.

\[
\tau_0 = t_K = T, \quad \tau_1 = \tau_{K-1}, \ldots, \tau_K = t_0 = 0
\]

The size of an intervall is \( \Delta \tau = T/K \) and the time grid points for backward computing becomes
\[ \tau_k = T - k\Delta\tau, \quad k = 0, 1, \ldots, K \]

Similarly we define the spatial grid points as follows

\[ S_i = i\Delta S, \quad i = 0, 1, \ldots, m \]

With the spacing and boundary points

\[ S_0 = 0, \quad S_m = Z/n, \quad \Delta S = \frac{Z}{nm} \]

Let \( V^k_i = V(S_i, \tau_k) \) be the approximative solution to (31) at the asset value \( S_i \) and time \( \tau_k \). Further we define \( V^k \) at the time \( \tau_n \) as follows.

\[ V^k = (V^k_0, V^k_1, \ldots, V^k_m)^T \]

We approximate the first time derivative with a forward difference by

\[ \frac{\partial V}{\partial \tau}(S_i, \tau_k) = \frac{V^{k+1}_i - V^k_i}{\Delta\tau} \]

For the first spatial derivative we use the central difference approximation as suggested in [L.Xingwen] for better convergence.

\[ \frac{\partial V}{\partial S}(S_i, \tau_k) = \frac{V^k_{i+1} - V^k_{i-1}}{2\Delta S} \]

The second spatial derivative becomes:

\[ \frac{\partial^2 V}{\partial S^2}(S_i, \tau_k) = \frac{V^k_{i+1} - 2V^k_i + V^k_{i-1}}{\Delta S^2} \]

The finite differences inserted in (31) gives us the following formulae

\[ V^k_i = -\alpha_i V^{k+1}_{i-1} + (1 + \alpha_i + \beta_i + r\Delta\tau)V^{k+1}_i - \beta_i V^{k+1}_{i+1} \]

Where \( \alpha_i \) and \( \beta_i \) are given by

\[ \alpha_i = \left( \frac{\sigma^2 S_i^2}{2\Delta S^2} - \frac{(r-D)S_i}{2\Delta S} \right) \Delta\tau \]

\[ \beta_i = \left( \frac{\sigma^2 S_i^2}{2\Delta S^2} + \frac{(r-D)S_i}{2\Delta S} \right) \Delta\tau \]
From the discretization we obtain a system of linear equations that can be expressed as a matrix equation. We define the matrix $D$ as follows.

$$
D = \begin{pmatrix}
-\tau & 0 & 0 & \cdots & 0 \\
\alpha_1 & -(\alpha_1 + \beta_1 + \tau) & \beta_1 & \cdots & 0 \\
0 & \alpha_2 & -(\alpha_2 + \beta_2 + \tau) & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \beta_m \\
0 & 0 & 0 & \cdots & 0 
\end{pmatrix}
$$

From the boundary condition in (9), $V(S, T) = Z$, we obtain the intitial vector of length $m + 1$:

$$V^0 = (Z, Z, \ldots, Z)^T$$

Then we can iterate through all the values of $V$ by the following matrix equation

$$(I - D)V^{n+1} = IV^n$$

The coupon payements are handled as follows: when we reach a time $t_c$ in our iteration where a coupon should be paid, the price is set to be:

$$V(S, t_c^+) = V(S, t_c^-) + K$$

The computation is done in Matlab and the code is to be found in Appendix C. We use the following values for our parameters:

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maturity date $T$</td>
<td>5 years</td>
</tr>
<tr>
<td>Interest rate $r$</td>
<td>5 %</td>
</tr>
<tr>
<td>Face value $Z$</td>
<td>100</td>
</tr>
<tr>
<td>Volatility $\sigma$</td>
<td>20%</td>
</tr>
</tbody>
</table>

In Figure 5 we can see how the value of the convertible bond behaves at $t = 0$, as one can tell the bond with coupon payment is worth more than one with zero coupon payment which is quite intuitive. In Figure 6 we have a plot in three dimensions which visualizes how the value of the convertible bond with zero coupon payment changes with time and stock price. We can see that it is monotone increasing in both the $t$-variable and the $S$-variable which we also proved in Section 4.3. In Figure 7 you can see how an annually coupon payment affects the value of the convertible bond, it drops immediately after the coupon payment and then rises again. In Figure 8
we have visualized the obstacle which separates the conversion region from the continuation region. As one can tell from the plot, the optimal stopping boundary $S(t)$ is convex.
Figure 5. Price of convertible bond as function of the stock price at $t = 0$

Figure 6. Price of convertible bond with zero-coupon payment
Figure 7. Price of callable convertible bond with coupon payment annually

Figure 8. Plot of the obstacle $\partial \{V \geq nS\}$, the blue area is the conversion region and the red area is the continuation region.
ACKNOWLEDGEMENT

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We would also like to thank Sadna Sajadini for taking the time to help and guide us.
APPENDIX A: DEFINITIONS

Financial derivatives: A collective name for financial instrument e.g. options, futures, forwards and swaps. These are linked to a specific moment or time period in the future and their value depends on the value of an underlying asset e.g. shares, currencies, rates and commodities.

European call option: A contract with the following conditions: At a prescribed time in the future, known at the expiry date, the holder of the option may purchase a prescribed asset (the underlying asset), for a prescribed amount; the exercise price.

Put option: Allows the holder to sell the asset on a certain date for a prescribed amount and the writer is obliged to buy the asset.

Selling short: Selling shares you do not own, a fall in the shares will give you profit.

Hedging: Making an investment to reduce the risk of adverse price movements in an asset, e.g. possess both assets and puts of the same shares. Perfect hedging eliminates the risk.

American option: An option that may be exercised at any time before maturity. Can be interpreted as free boundary problems.

Bond: A bond is a type of loan to a company/government. An investor loans money to a company/government when it buys its bonds. It’s a debt investment in which an investor lends money to an entity. The funds are borrowed for a defined period, we have fixed interest rate (coupon).

Maturity date: date where the loaned funds must be returned (the bond obligation ends).

Secured/unsecured: unsecured (debentures), if company fails, might get little from investment back (only guaranteed by credit of issuing company). Secured: specific assets are promised to bondholders if company cannot pay.

Callability: The bond can be called back by the issuer before maturity. Often favors the issuer over the investor. The investors money is fully returned with a premium.

Different types of risks: CALL RISK, MARKET RISK, INTEREST-RATE RISK, CREDIT RISK

Bond ratings: Agencies rate a company’s ability to repay its debts, obligations. “High grade”, “investment grade”, “junk bonds”.

Convertible bond: A bond that can be exchanged for company stocks, converted to company stocks by the holder at a later date. A hybrid between a bond and call option. The conversion ratio is the number of shares that can be converted for each bond.

Volatility: measure of the rate and magnitude of the change of prices (up
or down) of the underlying asset. Variable in the BS model of option pricing. Statistical volatility. Implied volatility (number in the BS that makes a theoretical price match the market price).

**Concept of arbitrage**: when future price of an investment asset is unknown, we assume that it’s future market price is determined by the price of another asset whose future price is deterministic, for example amount of money one can get from putting it in a bank account with known interest rate.

**Ito’s lemma**: This lemma relates a small change in a function of a random variable to the change of the variable itself. If $f$ is a function of a random variable $G$ that follows a stochastic differential equation of the form

$$dG = A(G, t)dt + B(G, t)dX$$

then for a given $f(G)$ we have:

$$df = A \frac{df}{dG} dG + \left( B \frac{df}{dG} + \frac{1}{2} A^2 \frac{d^2f}{dG^2} \right) dt$$
Appendix B: The European option

To be able to determine a unique solution to the Black-Scholes equation we need to consider final and boundary conditions. For an European call option with value $C(S, t)$ the final condition i.e. the value at the expiry date $t = T$ is

$$C(S, t) = \max (S - E, 0)$$  \hspace{1cm} (32)

That is because the profit one will make from it is equal to $S - E$ for $S > E$ but if $E > S$ the profit will be zero. Futher we want to consider the cases when $S = 0$ and as $S \to \infty$. By considering equation (1) one can easily see that if $S$ is zero, then so is $dS$, and hence $S$ can never change. Therefore if $S = 0$ at expiry the payoff is zero and the call option is worthless. This can be written as the following boundary condition

$$C(0, t) = 0$$  \hspace{1cm} (33)

On the contrary if $S$ raises infinitely the option will most likely be exercised and the exercise price will be of no interest compared to the asset price. Thus the value of the option becomes that of the asset, hence

$$C(S, t) \sim S \quad \text{as} \quad S \to \infty$$  \hspace{1cm} (34)

(More accurately:

$$C(S, t) \sim S - E e^{-r(T-t)} \quad \text{as} \quad S \to \infty$$

The Black-Scholes equation with boundary and final conditions pursuant to (32), (33) and (34) can be solved exactly and thus we can price an European call option.

Let us continue by consider the boundary and final conditions for the value of a put option $P(S, t)$. Just like above, the final condition is equal to the payoff

$$P(S, T) = \max (E - S, 0)$$  \hspace{1cm} (35)

We know that if the value of $S$ is ever zero, then it is always zero, and thus the final payoff must be $E$. To determine the value, $P$, for $S = 0$ and various $t$, we must calculate the present value of $P$ which we know is $E$ at $t = T$. Considering constant interest rates it follows that
If the interest rate is time-dependent, the boundary condition conforms the following formulae

\[ P(0, t) = Ee^{-\int_t^T r(\tau) d\tau} \]

Finally we consider the case when \( S \to \infty \). For very large \( S \) the option is obviously improbable to be exercised and its value will equal zero.

\[ P(S, t) \to 0 \quad \text{as} \quad S \to \infty \]

The value at \( S = 0 \) and \( S = \infty \) is not necessary, but we need to know that the value of the option is not too singular.

Consider the Black-Scholes equation and boundary conditions for a European call option, \( C(S, t) \)

\[ \frac{\partial C}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS \frac{\partial C}{\partial S} - rC = 0 \]

\[ C(0, t) = 0 \]
\[ C(S, t) \sim S \quad \text{as} \quad S \to \infty \]
\[ C(S, T) = \max (S - E, 0) \]

The first thing to do is to eliminate the nonconstant terms \( S \) and \( S^2 \). In the same step we can make the equation dimensionless and turn it into a forward equation, by setting

\[ S = Ee^x \]
\[ t = T - \frac{1}{2} \sigma^2 \]
\[ C = Ev(x, \tau) \]
Thus we have
\[
\begin{align*}
\frac{dS}{dx} &= S \\
\frac{\partial \tau}{\partial t} &= -\frac{1}{2} \sigma^2 \\
\frac{\partial C}{\partial t} &= -\frac{1}{2} E \sigma^2 \frac{\partial v}{\partial \tau} \\
\frac{\partial C}{dS} &= \frac{E}{S} \frac{\partial v}{\partial x} \\
\frac{\partial^2 C}{dS^2} &= \frac{E}{S^2} \left( \frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial x} \right)
\end{align*}
\]

Inserted in (38) we obtain
\[
\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + \left( r \frac{1}{2 \sigma^2} - 1 \right) \frac{\partial v}{\partial x} - \frac{r}{2 \sigma^2} v
\]

By setting \( k = \frac{r}{2 \sigma^2} \) the equation becomes
\[
\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + (k - 1) \frac{\partial v}{\partial x} - kv
\]

For the initial condition we obtain
\[
v(x, 0) = \max (e^x - 1, 0)
\]

To solve equation (5) we use the following change of variables
\[
v(x, \tau) = e^{\alpha x + \beta \tau} u(x, \tau)
\]
for some constants \( \alpha \) and \( \beta \). The differential equation now becomes
\[
\beta u + \frac{\partial u}{\partial \tau} = \alpha^2 u + 2\alpha \frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} + (k - 1) \left( \alpha u + \frac{\partial u}{\partial x} \right) - ku
\]

One can see that this becomes the diffusion equation if we eliminate the \( u \) and the \( \partial u/\partial x \) terms. This can be done by setting
\[
\beta = \alpha^2 + \alpha(k - 1) - k \\
0 = 2\alpha + (k - 1)
\]

Solving for \( \alpha \) and \( \beta \) we obtain
\[ \alpha = -\frac{1}{2}(k - 1) \]
\[ \beta = -\frac{1}{4}(k + 1)^2 \]

This substituted into the equation gives

\[ v(x, \tau) = e^{-\frac{1}{2}(k-1)x - \frac{1}{4}(k+1)^2\tau}u(x, \tau) \]

Where \( u(x, \tau) \) satisfies the diffusion equation

\[ \frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} \quad \text{for} \quad -\infty < x < \infty, \quad \tau > 0 \]

Let us assume the interest rate, \( r \), and the volatility, \( \sigma \) constant. Then we have the exact solution for the European call as

\[ C(S, t) = SN(d_1) - Ee^{-r(T-t)}N(d_2) \]

where \( N(\cdot) \) is the cumulative distribution function for a standardised normal random variable, given by

\[ N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}y^2} dy \]

We have \( d_1 \) and \( d_2 \) as follows

\[ d_1 = \frac{\log (S/E) + (r + \frac{1}{2}\sigma^2)(T - t)}{\sigma \sqrt{T - t}} \]
\[ d_2 = \frac{\log (S/E) + (r - \frac{1}{2}\sigma^2)(T - t)}{\sigma \sqrt{T - t}} \]

For a put we have the solution

\[ P(S, t) = Ee^{-r(T-t)}N(-d_2) - SN(-d_1) \]
Appendix C: MATLAB code

clc;
clear all;
close all;

dt=0.1;
dS=1;
T=5;
Z=100;
n=1;
D=0.01;    %dividend
S_max=100  %S_max=150;
sigma=0.2; %volatility
r =0.05;   %interest rate
K=4;       %coupon payment
K_int =[1,2,3,4];
t=T:-dt:0;
S=0:dS:S_max;

alpha=((sigma^2*S.^2)/(2*dS^2) - ((r-D) *S)/(2*dS))*dt;
beta=((sigma^2*S.^2)/(2*dS^2) + ((r-D) *S)/(2*dS))*dt;

alpha2=alpha;
alpha2(1)=[];
alpha2= [alpha2 0];

d= -(r*dt + alpha + beta) 0;
Di = diag(d) + diag(beta,1) + diag(alpha2,-1);

V=Z*ones(1,length(Di));
V=V';

V_tot=[];
for i=0:dt:T
    V=(eye(length(Di))-Di)
    in = ismember(i,K_int);
    if in == 1
        V=V+K*ones(length(V),1);
    end
end
for j=1:length(V)-1
    if V(j)>Z
        V(j)=n*S(j);
    end
    if V(j)<n*S(j)  %kollar obstacle
        V(j)=n*S(j);
    end
end
for j=1:length(V)-1
    V(j)=max(V(j),n*S(j));
end
V_tot=[V_tot V];
end

V=Z*ones(1,length(Di));
V=V';
V_0=[]
for i=0:dt:T
    V=(eye(length(Di))-Di)V;
    in = ismember(i,K.int);  %coupon payment
    if in == 1
        hej= 1
        V=V+K*ones(length(V),1);
    end
    for j=1:length(V)-1
        if V(j)<n*S(j)  %kollar obstacle
            V(j)=n*S(j);
        end
    end
    V_0=[V_0 V];
end
t(length(t))= [];
V_2=size(V_tot);
V_tot(:,V_2(2))=[];
V_tot(V_2(1),:)=[];
V_tot=V_tot';
V_0(:,V_2(2))=[];
V_0(V_2(1),:)=[];
V_0=V_0';
plot(S',V_tot(V_{2}(2)-1,:), 'r')
hold on
plot(S',V_0(V_{2}(2)-1,:))
xlabel( 'Stock Price')
ylabel('Convertible Bond Value')
figure
legend('Coupon payment convertible bond','Zero-coupon convertible bond')
[X,Y] = meshgrid(S,t);
figure
colormap([0.5 0.5 0.5])
mesh(X,Y,V_tot);
xlabel('Stock Price')
ylabel( 'Time ')
zlabel ( 'Convertible Bond Value ')
REFERENCES


