Development of resistive MHD code in cylindrical geometry and its applications on EXTRAP T2R

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Master Thesis
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Erasmus Mundus Program on Nuclear Fusion Science and Engineering Physics
Abstract

A resistive magnetohydrodynamic (MHD) code is presented in detail for a cylindrical plasma column surrounded by a perfect conducting wall. The objective is to develop a full eigenvalue problem solver with resistive wall type boundary conditions that can be integrated into the feedback control algorithms of the EXTRAP T2R reversed field pinch (RFP).

For the straight tokamak model, linear analysis is carried out around a flowless background equilibrium followed by the normal mode expansion. The numerical method here applied relies on the weak formulation of the Galerkin method. Its implementation gives rise to a general eigenvalue problem which concerns the discretization of non-self adjoint matrix operator. Hence, we are forced to consider all variables to describe the dynamics of the system.

Simulations were performed for both ideal and resistive MHD models. In the former case, the results show that the complete ideal spectrum can be extracted accurately, viz. the magnetoacoustic waves, Alfvén and slow modes, provided a shareless background magnetic field and in presence of the \( (m, n) = (2, 1) \) interchange instability. Particularly, the behaviour near the marginal point \( \omega \to 0 \) is in agreement with theoretical predictions. We emphasize the subtleties regarding the optimal choice of space discretization, showing that the use of quadratic and cubic finite elements avoids numerical pollution.

Concerning the resistive MHD calculations, the effect of resistivity is studied for a tokamak-like equilibrium profiles. Results show that the two most unstable modes, namely the tearing and interchange instabilities, can be extracted from the code simultaneously. Moreover, the scaling relations with resistivity for both growth rates are presented, showing the \( \propto \eta^{3/5} \) and \( \propto \eta^{1/3} \) dependence for the tearing and interchange modes, respectively. Nevertheless, the results are found to be valid only for small values of resistivity. In addition, the effect of resistivity on the perturbed profiles is presented. The foreseen extension of the code to study external modes is discussed.
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Chapter 1

Introduction

1.1 Energy outlook in the 21st century

Energy scarcity and environmental pollution are two critical matters that human kind faces in the present century. To overcome these issues, several attempts have been done by means of international treaties [1], new energy policies together with renewable energy action plans [2, 3], and research & development roadmaps. However, all the joint efforts are threatened since, on the one hand the world energy demand is expected to increase rapidly in the next 40 years in correlation with the global population growth, whereas on the other hand the currently most used energy sources are finite. Figure 1.1 shows the perspective of the primary energy sources use for the next decades.

**Figure 1.1**: Global perspective on primary energy use: 2000-2100 [4].
Therefore, with the above mentioned challenges it is more than obvious that both fundamental and applied research need to examine a wide spectrum of energy resources and their possible use. An interesting approach to this can be found in [5], where the following 4 points are proposed:

1. Identify potential energy sources which minimizes greenhouse gas emissions and that meets high-efficiency energy technologies.

2. Identify possible barriers for large-scale applications of these technologies.

3. Conduct fundamental research into technologies that will help to overcome these barriers and provide a basis for large-scale applications.

4. Share research results with a wide audience, including the science and engineering community, media, business, governments, and potential end-users.

Furthermore, the 4 points are directly related to both population growth and energy consumption projections. Bearing in mind these two additional factors, now the question is, among all possible energy sources, which one fulfills the world’s demand in a short or a mid-term? Before we answer it, only by looking at figure 1.2 we can have an idea of the tremendous amount of the already required energy by 2006, where the highly desirable scenario is the one considering a substantial consumption’s decrease (orange and red arrows).

\[\text{Figure 1.2: The per capita income for all countries with more than 20 million inhabitants -more than 90% of the world’s population- as a function of the per capita average energy consumption rate (for the year 2006) [6, 7].}\]
However, the reality is yet another one since the energy demand rises up every year [8, 9]. Staying with the above case, in 2006 the total worldwide energy consumption was about $5 \times 10^{17}$ BTU ($1 \text{ BTU} = 1.055 \times 10^3 \text{ J}$). For comparison, the total energy from the Sun reaching the Earth’s surface in one year is about $3 \times 10^{21}$ BTUs. For instance, the International Energy Agency predicts that energy use will increase 60% by 2030 and double by 2045. Currently, 80% is derived from burning fossil fuels such as coal, oil, and gas which are the main malefactor in global climate change and pollution. Moreover, eventually the associated reserves will run out making the final-end products unaffordable for the average costumers. For this reason, long-term strategies, which may exploit renewable energy and suppress the CO$_2$ emissions, are still needed and this of course will not be an easy task to achieve. For example, even in the more optimistic 40-year scenario in emerging markets such as the chinese, assuming both the discontinuity of burned fossils and an increase of renewables energy sources, the latter will not meet more than 20% of China’s total energy production [10]. In addition, in [10] it is also stated that despite of 15% (upper limit) of the total energy production by means of nuclear fission, both renewable and fission energy sources will not suffices the foreseen chinese demand.

![Figure 1.3](image-url)

**Figure 1.3:** Inventory of the energy solutions for the present and future. Adapted from R. Wiejermars et al. [11].
1. Introduction

Thus, after all, where do we stand? Figure 1.3 resumes the actual and foreseen solutions to the global energy problem by characterizing the energy potential (of energy sources) with time \[11\]. From the plot we can extract the importance of developing, as a short-term goal, new technologies which provides a basis of large-scale applications of new energy sources. Because, even if fossil fuels are left out and a significant boost in renewable energy rises, still a large energy demand gap needs to be filled out, and thereby humankind will face severe energy problems in the up coming years. However, for our relief there is a very attractive energy source which is secure, environmental friendly, and it is regarded as one of the few options that could meet future’s energy demand: fusion energy.

1.2 Fusion Energy

The attractiveness of nuclear fusion relies on obtaining high energy gain by fusing light elements, typically helium-3 (\(^3\)He) and both hydrogen isotopes deuterium (D) and tritium (T). Regarding D and T as primary fuels, another potential attraction is that both D and T Earth’s reserves are almost inexhaustible. D is found naturally in sea water, whereas T is obtained by processing Li\(^1\) which can be found in rocks\(^2\). According to \[12\], the estimated world’s Li reserves are equivalent to 1775 years of supply at current rate of demand (2008). In table 1.1 it is shown a list of possible fusion reactions.

<table>
<thead>
<tr>
<th>Fusion reactions</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>D + T</td>
<td>(\rightarrow) (^4)He (3.5 MeV) + n (14.1 MeV)</td>
</tr>
<tr>
<td>D + D (50%)</td>
<td>(\rightarrow) T (1.01 MeV) + p (3.02 MeV)</td>
</tr>
<tr>
<td>D + (^3)He (50%)</td>
<td>(\rightarrow) (^3)He (0.82 MeV) + n (2.45 MeV)</td>
</tr>
<tr>
<td>D + (^3)He (51%)</td>
<td>(\rightarrow) (^4)He (3.6 MeV) + p (14.7 MeV)</td>
</tr>
<tr>
<td>T + T</td>
<td>(\rightarrow) (^4)He + 2n (14.1 MeV) + 11.3 MeV</td>
</tr>
<tr>
<td>T + (^3)He (43%)</td>
<td>(\rightarrow) (^4)He + p + n + 12.1 MeV</td>
</tr>
<tr>
<td></td>
<td>(43%) (\rightarrow) (^4)He (4.8 MeV) + D (9.5 MeV)</td>
</tr>
<tr>
<td></td>
<td>(6%) (\rightarrow) (^5)He (2.4 MeV) + D (11.9 MeV)</td>
</tr>
<tr>
<td>p + (^{11})B</td>
<td>(\rightarrow) (^3)(^4)He + 8.7 MeV</td>
</tr>
<tr>
<td>n + (^6)Li</td>
<td>(\rightarrow) (^4)He (2.1 MeV) + T (2.7 MeV)</td>
</tr>
</tbody>
</table>

Table 1.1: Fuel cycles (fusion reactions) \[14\] (branching ratios are correct for energies near cross section peaks; a positive yield means the reaction is exothermic, otherwise endothermic).

\(^1\)Li stands for lithium. Strictly speaking, T is obtained from the breeding reaction between lithium isotope \(^6\)Li and neutron.

\(^2\)Actual and potential sources of lithium are: pegmatites, continental brines, geothermal brines, oilfield brines and the clay mineral hectorite \[12, 13\].
1.2 Fusion Energy

To trigger a fusion reaction, two nuclei have to move at sufficient relative velocity, so that the Coulomb barrier can be overcome. Such situation is made evident in figure 1.4 where the repulsive potential is Coulombian,

\[
V_C(r) = \frac{1}{4\pi\epsilon_0} \frac{Z_1 Z_2 q_e^2}{r},
\]

at distances greater than \( r_n \approx 1.44 \times 10^{-13} \left( A_1^{1/3} + A_2^{1/3} \right) \) cm,

where \( \epsilon_0 \) is the vacuum permittivity, \( Z_1 \) and \( Z_2 \) are the atomic numbers, \( A_1 \) and \( A_2 \) are the mass numbers of the interacting particles, and \( q_e \) is the electron charge.

**Figure 1.4**: A repulsive Coulomb potential acts on the particles at distances greater than \( r_n \), whereas a nuclear potential well \(-U_0\) attracts both particles at distance lower than \( r_n \). To achieve the latter a Coulomb barrier \( V_b \) should be overcome (of the order of 1 MeV) \[15\].

On the contrary, at smaller distances than \( r_n \) the two nuclei are attracted by means of a nuclear potential well \(-U_0\) with a typical magnitude around 30-40 MeV\(^4\) \[15, 16\]. As mentioned earlier, only particles with sufficient relative energy can overcome the barrier \( V_b \), otherwise there exists a relative energy threshold \( \epsilon < V_b \) that allows the particles to approach each other up the

\(^3\)Other parameters such as cross section and rate of fusion reactions are crucial for fusion to occur.

\(^4\)1 eV = 1.602 \times 10^{-19} \text{ J} \text{ and } 1 \times 10^6 \text{ eV} = 1 \text{ MeV}.
classical turning point \( r_{tp} \)

\[
r_{tp} = \frac{1}{4\pi \epsilon_0} \frac{Z_1 Z_2 q^2}{\epsilon}.
\]

However, quantum mechanics states that fusion reactions are indeed possible via tunneling, viz. particles can cross the Coulomb barrier with a probability different from zero. It is the fusion cross section \( \sigma_{\text{fus}} \) that gives the probability for fusion to occur and usually it is written as [15, 17]

\[
\sigma_{\text{fus}} = \frac{S(\epsilon)}{\epsilon} \exp \left( -\sqrt{\frac{\epsilon G}{\epsilon}} \right) \text{barn},
\]

where 1 barn = \( 10^{-24} \) cm\(^2\) and \( \epsilon \) is the relative energy between D and T as before. The factors \( s(\epsilon), \epsilon, \exp \left( -\sqrt{\frac{\epsilon G}{\epsilon}} \right) \) are the astrophysical (S-factor), geometric factor and Gamow factor, respectively. The S-factor is a smooth function of the relative energy \( \epsilon \) and contains the nuclear information of the system under consideration, whereas the geometric factor is essentially related to the de Broglie wavelength of the system. The Gamow factor represents the dependence of the transition probabilities due to the tunnel effect. In the latter factor, \( \epsilon_G \) is called the Gamow energy which is an intrinsic characterization of the barrier [18] and depends quadratically on the atomic number of both nuclei and the reduced mass of the system\(^5\).

Moreover, for reactor studies it is the number of fusion reactions per unit volume and unit time that we want to evaluate. Because knowing this rate, we can actually obtain the power emitted by fusion reaction, and from this an eventual electricity production is envisaged. The rate per unit volume \( R \) involving nuclei 1 and 2 is given by [18]

\[
R = n_1 n_2 \langle \sigma_{\text{fus}}(\epsilon)\epsilon \rangle,
\]

where the average of \( \sigma_{\text{fus}}(\epsilon)\epsilon \) is taken over the velocity space (\( v \propto \sqrt{\epsilon} \)) at a given temperature \( T \). Thus, via \( R \) and the fusion energy density \( E_{\text{fus}} \) per reaction, the fusion power density \( P_{\text{fus}} \) is defined as

\[
P_{\text{fus}} = E_{\text{fus}} R.
\]

Recalling the table 1.1, there are two main reactions of interest that occur at a fast enough rate to eventually produce electricity, namely the D-T (50\% - 50\% mix) and pure D-D reaction. The one involving D-T is the most promising for fusion reactor assessments, because its largest fusion cross section (peaking \( \sim 5 \) barns) and its lowest energy in which the maximum is reached [19] (\( \sim 65 \) keV in the center-of-mass). This is compared to a maximum of 0.819 barns

\(^5\)According to [15, 18] the Gamow energy is given by \( \epsilon_G = 2\mu (\pi Z_1 Z_2 \alpha c)^2 \text{keV} \). Where \( \mu \) is the reduced mass of the system, \( \alpha \) is the fine-structure constant and \( c \) is the speed of light in vacuum.
at 262 keV for $^3$He-D and 1.2 barns at 600 keV for p-$^{11}$B reaction (see figure 1.5). Therefore, we will focus in the following fuel cycle

\[ \text{D} + \text{T} \rightarrow ^4\text{He (3.5 MeV)} + \text{n (14.1 MeV)}. \]

$^4$He will be referred as the $\alpha$-particles in what follows. Once the fuel cycle is selected, the next step is to know what are the minimum requirements for a fusion reactor to meet the ignition condition. This condition could lead a self-sustainable operation mode.

**Figure 1.5**: Fusion cross sections for D-T, D-D, $^3$He-D and p-$^{11}$B [19].

### 1.2.1 The triple product

To develop the minimum requirements for an ignition condition, we need to consider what are the energy sources feeding the system on the one hand, and on the other hand we need to determine the sources that provide energy losses. To do so, first we define our system to be a fusible matter of volume $V$ consisting of 50%-50% mixture of D-T, with negligible concentration of $\alpha$-particles, i.e.

\[
2n_D = 2n_T = n_e \equiv n, \quad n_\alpha \ll n, \quad \text{(1.5, 1.6)}
\]
where $n_j$, is the number density of the $j$-th species of particles and the condition \( \sum_j n_j = n_e \) is used\(^6\). Now, the temporal variation of the total energy density of system $u_{tot}$ is determined by both the gain (energy $> 0$) and the losses (energy $< 0$)

\[
\frac{d}{dt} u_{tot} = P_{\text{Gain}} - P_{\text{Loss}},
\]

\[
= (\eta_{\text{eff}} P_{\text{fus,}\alpha} + P_h) - (P_{L,\kappa} + P_{L,B}),
\]

(1.7)

here $P_{\text{fus,}\alpha}$ is the fusion heating power density provided by the $\alpha$-particles\(^7\) and its energy deposition efficiency $\eta_{\text{eff}}$. $P_h$ is the external heating power density supplied to the system (typically by ohmic heating power or RF power), whereas $P_{L,\kappa}$ is the power density loss by transport processes (conduction and convection) and $P_{L,B}$ the power density loss by radiation (Bremsstrahlung).

In an steady state regime, the temporal variation of $u_{tot}$ in equation (1.7) vanishes, thus leading to the following power balance relation

\[
\eta_{\text{eff}} P_{\text{fus,}\alpha} + P_h = P_{L,\kappa} + P_{L,B}.
\]

(1.8)

The above power balance is the starting point to develop a criterion for having fusion in a reactor. However, we need to introduce one figure of merit that qualifies the system’s power balance: the amplification factor $Q$. This is defined as the ratio between the power from fusion reactions $P_{\text{fus}}$ and the external power supplied to the system by the heating sources $P_h$:

\[
Q = \frac{P_{\text{fus}}}{P_h},
\]

(1.9)

if $Q > 1$, more energy has been produced with fusion reactions than was necessary to supply the system. $Q = 1$ is the so-called Break-even situation, both fusion power equalizes the external heating power. Finally, the Ignition regime is the situation where the power supplied by the fusion reactions is enough on its own to compensate losses, this corresponds to an infinite $Q$ amplification factor ($S_h = 0$).

We now return to the power balance equation. The power loss $P_{\text{Loss}}$ will be associated to the decrease of the system’s internal energy density $W$. The latter can be expressed in terms of the global temperature $T$\(^8\) by means of the equipartition energy theorem

\[
W = \sum_{j=\{D,T,e\}} W_j = \sum_{j=\{D,T,e\}} \frac{3}{2} n_j T_j,
\]

\[
= \frac{3}{2} (n_D + n_T + n_e) T,
\]

\[
\text{Eq.}(1.5) \leftarrow 3nT,
\]

(1.10)

\(^6\)Also known as a quasi-neutrality condition which will be developed in the next chapter along with the general description of a plasma.

\(^7\)The total power produced by the D-T fusion reaction is divided between the products of the reaction, viz. the $\alpha$ particles and the neutrons. This gives: $P_{\text{fus}} = \eta_{\text{eff}} P_{\text{fus,}\alpha} + P_{\text{fus,n}}$.

\(^8\)It is assumed that the system is in thermal equilibrium, i.e. $T_D = T_T = T_e \equiv T$. 
1.2 Fusion Energy

where $T$ is given in eV\(^9\) and it is related to the system’s total pressure by $p = 2nT$. In addition to this, it is introduced a characteristic time in which the energy content of the system decreases and it concerns the quality of the thermal insulation of the confinement scheme. This quantity is called the energy confinement time $\tau_E$, defined by

$$
\tau_E \equiv \frac{W}{P_{\text{loss}}} = \frac{3nT}{P_{L,\kappa} + P_{L,B}},
$$

(1.11)

Typically, $\tau_E$ is determined experimentally by regression analysis of large database available for fusion devices. Introducing (1.9) and (1.11) into (1.8) together with the assumption that only the total fusion power is deposited into the system by the $\alpha$-particles\(^10\). Then, the power balance reads

$$
\frac{3nT}{\tau_E} = \eta_{\text{eff}} (1 + Q^{-1}) P_{\text{fus},\alpha},
$$

(1.12)

we recall the definition of the fusion power density $P_{\text{fus}}$ (see Eq. (1.4)) together with the 50%-50% of D-T mixture assumption of Eq. (1.5), then the fusion power density $P_{\text{fus},\alpha}$ is given by

$$
P_{\text{fus},\alpha} = \frac{1}{4} n^2 \langle \sigma v \rangle E_{\text{fus}},
$$

(1.13)

introducing this into Eq. (1.12) followed by some algebra, it is obtained

$$
n\tau_E T = \frac{12T^2}{\eta_{\text{eff}} (1 + Q^{-1}) \langle \sigma v \rangle E_{\text{fus}}},
$$

(1.14)

This is the so-called triple product which expresses the constraint on the plasma parameters (density, temperature and confinement time of the energy). From the above expression, the minimum condition for ignition reads

$$
n\tau_E T > 3 \times 10^{21} \text{m}^{-3} \text{keV s},
$$

(1.15)

which takes place at a temperature about 15 keV (150 mill. °C). This seems a very hard task to achieve. However, an attempt to do this considers to shape and confine D-T in a given volume inside a furnace chamber of reasonable size (several meters across) by applying external magnetic fields (several Tesla). This corresponds to the so-called magnetic confinement fusion (MCF) approach. The MCF devices are characterized to have toroidal geometry, in order to avoid end losses. There are mainly three branches in MCF research, namely the tokamak, stellarator and reversed-field pinch (RFP).

\(^9\)1 eV = $10^4$ K.
\(^{10}\)It is only considered the energy remaining within the system (neutrons are leaving the system), viz. $P_{\text{fus}} \rightarrow \eta_{\text{eff}} P_{\text{fus},\alpha}$.
Additionally to the triple product, another figure of merit called the \( \beta \) parameter is widely used in MCF. The importance of \( \beta \) is that it is a measure of the thermonuclear power obtained for a given magnetic field strength. Its value is given in by

\[
\beta = \frac{\langle p \rangle}{B_T^2/2}
\]  

(1.16)

where \( \langle p \rangle \) is the average plasma pressure and \( B_T \) is total magnetic field which confines the system; \( \beta \) has no dimensions. The actual challenge in plasma physics and fusion reactor studies relies on the achievement of the optimal combination between \( T \), \( \tau_E \), and \( \beta \), simultaneously. It turns out [20] that \( \beta \approx 8\% \), \( \tau_E \approx 1 \) s and \( T \approx 15 \text{ keV} \) are the critical values for ignition to occur.

### 1.2.2 Status quo and the fusion reactor era

![MCF branches: tokamaks, stellarators and RFP. The triad: experiment, theory and simulation are crucial on the development towards the fusion power generation era.](image)

Although nuclear fusion is unlikely to be ready for commercial power generation in the next couple of decades, fusion research has been constantly developed and it has achieved significant breakthroughs. For instance, more than 16 MW fusion power has been produced in JET [21], the world largest tokamak located in Culham in the UK. Equivalent \( Q \) is over unity in the JT-60U Japanese tokamak [22]. Moreover, high performance hot plasma has been kept for more than 0.5 h in the Large Helical Device Japanese stellarator [23, 24].
The next step towards the path to nuclear fusion energy is the International Thermonuclear Experimental Reactor (ITER), an international joint project under construction in Cadarache, south France.

ITER would provide the feasibility of nuclear fusion as a new energy source. It has been designed to produce 500 MW of output power providing solely 50 MW input power, i.e. an amplification factor $Q$ of 10. Furthermore, towards the fusion power generation, ITER will provide the necessary knowledge for design and technology assessment of the next fusion device: DEMO. A conceptual design for such a machine could be completed by 2017 [25]. It is forecast that DEMO will begin operation in the early 2030s, and tentatively putting fusion power into the grid as early as 2045, see figure 1.7.

![Figure 1.7: The Fast Track towards industrial fusion era. The model has been developed by UKAEA at Culham, United Kingdom [6].](image)

However, it should be noted that it is possible that fusion energy will not be on the energy market by the end of 21st century if fusion R&D does not go as planned or its economic efficiency and reliability of operation are not good enough. Thus, tremendous tasks need to be accomplished to reach fusion power generation. For instance, scientific community has devised the challenges that fusion researchers should overcome. Hence, towards an eventually commercial fusion power era, the current forefront of MCF research focus on the following subjects:

11R&D stands for research and development.
12Economical and financial crisis directly affect applied and fundamental research programs.
1. Introduction

- Tokamak, stellerator and RFP physics.
- Plasma-wall interaction.
- Materials research.
- Diagnostics.
- Data analysis (adquisition, real-time analysis).
- Plasma control (feedback control, MHD instabilities, heating and vacuum systems).

1.3 Outline

Confinement is a fundamental issue for thermonuclear fusion to happen. Nevertheless, changes in pressure and temperature in the system gives rise to unstable behaviour degrading the global parameters that affects the confinement. The instabilities, which are described by the magnetohydrodynamic (MHD) model, have a bearing on the system performance and the safety matters of the machine. Since the quality of confinement relies strongly on the mitigation of these instabilities, the operational regime of the fusion devices -tokamak, stellerator or RFP- should concern the most dangerous conditions for the global parameters.

At present, in order to enhance the plasma confinement [26], active feedback control is used by managing local perturbations of the magnetic configuration. Moreover, active feedback control systems are applied to control global plasma parameters such as the plasma position and shape, thus avoiding unstable situations due to elongation [26, 27].

Apart from position and elongation issues, there are potential benefits to devise an “intelligent” boundary magnetic feedback algorithms that modify the resistive MHD stability of the fusion machines. Recent work in RFP’s boundary feedback control that includes both ideal and resistive MHD modes is reported in [28, 29]. To study this in detail a full eigenproblem solver is required. A relevant point to develop such solver is to consider a general resistive wall type boundary conditions in which the feedback relations can be inserted. This work delves into the development and analysis of a resistive eigenvalue problem solver for potential applications on EXTRAP T2R\textsuperscript{13} RFP’s MHD boundary feedback methods.

Since this work deals with plasma physics, Chapter 2 provides the basic definitions of the plasma state, and gives a general overview of three theoretical models which describe plasma dynamics, namely the single particle picture, kinetic theory and fluid description of the plasma.

\textsuperscript{13}KTH’s in-house reversed field pinch, located at Alfvén laboratory (Stockholm).
1.3 Outline

The cylindrical resistive MHD model is usually required to describe the main RFP dynamics\textsuperscript{14}[30]. Therefore the first part of Chapter 3 introduces the basic resistive MHD equations followed by an equilibrium and stability analysis. Based on this, linear analysis is performed in the framework of the normal mode expansion which allows us to describe perturbations in the plasma.

The second part of Chapter 3 concerns the numerical model of the cylindrical plasma column. It gives an insight of the domain discretization technique and approximate solution via the Finite Element Method (FEM) as well. FEM is the basis of the Galerkin method (its weak formulation), which is the core of our numerical approach, it is thus revealed the basic mechanisms of the Galerkin method. Furthermore, both boundary conditions (interfaces) and solution expansion functions are discussed as well.

The application of the code is shown in Chapter 4, where the results for ideal and resistive MHD are presented and further analyzed. It is also discussed the distinct algorithms which provide solutions to large matrix problems and the limitations of the model. Finally, Chapter 5 presents the summary, conclusions and final remarks of the work.

\textsuperscript{14}At least basic resistive MHD is required to address resonant instabilities, since ideal MHD does not describes completely the plasma dynamics.
1. Introduction
Chapter 2

Plasma basics

Plasma is an ionized gas which contains free electrons, ions and neutrals. It is characterized for being globally neutral and by its collective behaviour provided electromagnetic forces that couples charged particles. Plasma state concerns matter that is ionized to a certain degree, thus in thermodynamic equilibrium, the ionization degree is expressed by the Saha equation [31]

\[
\frac{n_i}{n_n} \approx 3 \times 10^{27} \frac{T^{3/2}}{n_i} e^{-E_{ion}/T},
\]

where \(n_e, n_i\) and \(n_n\) are the mass densities (in \(m^{-3}\)) of electrons, ions and neutrals, respectively. The temperature \(T\) is given in eV\(^1\) and \(E_{ion}\) is the ionization energy of the ion.

In nature it is found different types of plasmas and their classification relies on the ionization degree and the value of the coupling parameter

\[
\Gamma_e = \frac{E_{pot}}{E_{kin}},
\]

here \(E_{pot}\) stands for the electrostatic interaction energy and \(E_{kin}\) the mean thermal energy. For instance, the ideal plasma condition is given by

\[
\Gamma_e \ll 1 \quad \Rightarrow \quad \frac{E_{pot}}{E_{kin}} = \frac{q_e^2}{4\pi\epsilon_0d} \frac{1}{3/2T_e} \ll 1,
\]

where \(q_e\) is the electron charge, \(\epsilon_0\) vacuum permittivity, \(T_e\) the electron temperature and \(d\) the average distance of two particles. This relation is particularly interesting, since most of the plasma in nature is in ideal state.

However, the ideal plasma condition is valid until certain limit (as always suspected with ideal conditions). For example, if the thermal energy fulfills \(T_e \geq m_{0e}c^2\), where \(m_{0e}\) is the electron’s rest mass and \(c\) is the speed of light in vacuum, then we are in the relativistic plasma regime. Also, one finds the so-called degenerated plasma regime (either relativistic or non-relativistic) if the following condition holds \(T_e \lesssim E_F\), where \(E_F\) is the Fermi energy.

\(^1\)1 eV \(\approx 1000\) K. The Boltzmann constant is absorbed in the temperature and thereby omitted.
2.1 Plasma properties

Debye shielding

Plasma has the capacity to shield every charge in the plasma with a cloud of charge with the opposite sign, thus screening it. This is the Debye shielding. By equating the potential energy of charge separation with the kinetic particle energy one finds the Debye length $\lambda_D$, which defines a typical length scale in plasmas

$$\lambda_D = \sqrt{\frac{\epsilon_0 T}{n q_e^2}}, \quad (2.4)$$

given in m. As before $\epsilon_0$ is the permittivity in vacuum, $T$ temperature, $n$ density and $q_e$ electron charge. If $L$ is the size of an ionized gas, then it is considered as plasma if $L \gg \lambda_D$.

Plasma parameter

Denoted by $N_D$, the plasma parameter describes the number of particles inside a sphere of radius $\lambda_D$ (also known as Debye sphere) by means of

$$N_D = n \frac{4}{3} \pi \lambda_D^3, \quad (2.5)$$

for an ideal plasma it holds $N_D \gg 1$.

Quasi-neutrality

A plasma is quasi-neutral if

$$n_e = \sum_j Z_j n_{i,j}, \quad (2.6)$$

here $Z_j$ is the charge of the $j$-th ion specie. In the case of ion charge $Z = 1$, the quasi-neutrality condition reads $n = n_e = n_i$ (which is the case of an hydrogen plasma).

Plasma frequency

Regarded as a representative example of the collective behaviour of the plasmas, the plasma frequency $\omega_p$ tells us how fast the plasma can shield deviations from the quasi-neutrality condition. The oscillation frequency in the plasma is given by

$$\omega_{p,\alpha} = \sqrt{\frac{n_\alpha q_{\alpha}^2}{\epsilon_0 m_\alpha}}, \quad \alpha = e, i, \quad (2.7)$$

where $m_\alpha$ is the mass of the specie under consideration. The remaining factors were already defined.
2.2 Plasma physics description

The theoretical description of plasma processes is found in many flavours and depending on the type of phenomenon to describe, one may choose the model that best fits specific necessities. Generally speaking, three theoretical models are mainly used in plasma physics phenomena, namely:

(a) **Single particle motion.** The motion of a charged, non-relativistic particle under the influence of electric and magnetic fields is described by Newton’s equation of motion.

\[
m \frac{dv}{dt} = q \left( E + v \times B \right),
\]

(2.8)

where both the electric and magnetic field are prescribed and fulfill Maxwell’s equations. The development of the well-known guiding center theory follows from the solution of the above equation. Then, the drift velocity pops up which allow us a better understanding of particle confinement in laboratory fusion plasmas. One should be aware the arduous amount of theoretical work involved in this approach, which I will not develope here, classic reference of guiding center theory can be found in [32] and for a more recent comprehensive treatise [33, 31].

(b) **Kinetic theory.** Since a plasma consists of a very large number of interacting particles, an statistical approach is required. The task is to describe the collective behavior of the many charged particles that constitute the plasma by means of particle distribution function \( f_{e,i}(r,v,t) \) together with the methods of statistical mechanics. A survey into some basic kinetic concepts is relegated to the Appendix B.

(c) **Fluid theory (MHD).** Describes plasmas in terms of averaged macroscopic functions of \( r \) and \( t \). Specifically, fluid moments are calculated from the kinetic approach and a number of assumptions are made in order to obtain closure of the resulting system of partial differential equations (PDEs). From the calculated fluids moments, the so-called two-fluid description of the plasma pops out. It describes the dynamics for both ions and electrons. However, further simplifications can be done if it is introduced the velocity of the center of mass, the fact that the ion’s mass is much greater than the electron’s mass and the quasi-neutrality condition. If this is considered, then we are dealing with the single-fluid model of the plasma.

Since all the theoretical machinery in this work relies on the single fluid MHD model, it is worth to stress a few remarks on it.
2. Plasma basics

2.2.1 A few remarks on the ideal MHD model

The ideal MHD equations\(^2\) describe the motion of a perfectly conducting fluid interacting with electromagnetic (EM) fields. Maxwell’s equations describe the evolution of such fields in response of a current density \( j(\mathbf{r},t) \) and space charge \( \varrho(\mathbf{r},t) \)

\[
\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad (2.9) \\
\nabla \times \mathbf{B} = \mu_0 j + \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t}, \quad c \equiv (\varepsilon_0 \mu_0)^{-1/2}, \quad (2.10) \\
\n\nabla \cdot \mathbf{E} = \frac{\varrho}{\varepsilon_0}, \quad (2.11) \\
\n\nabla \cdot \mathbf{B} = 0. \quad (2.12)
\]

To complete the description of the plasma dynamics we need to relate the EM variables with the mass density \( \rho \) and the pressure \( p \) of the fluid. Therefore, we invoke the equation of gas dynamics together with the equation of motion for a fluid element

\[
\frac{\partial p}{\partial t} + \mathbf{v} \cdot \nabla p + \gamma p \nabla \cdot \mathbf{v} = 0, \quad (2.13)
\]

\[
\rho \left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \mathbf{v} = -\nabla p + \rho \mathbf{g} + \mathbf{j} \times \mathbf{B} + \varrho \mathbf{E}, \quad (2.14)
\]

\[
\mathbf{E}' \equiv \mathbf{E} + \mathbf{v} \times \mathbf{B} = 0, \quad (2.15)
\]

the last equation dictates the vanishing condition for the electric field in the plasma in a co-moving frame of reference. Next, some assumptions are made to further simplified the equations. The first one is to restrict our analysis to non-relativistic phenomena, i.e.

\[
v \ll c. \quad (2.16)
\]

This allow us to make an estimate for the order of magnitude of the different terms in Maxwell’s equations. Lets consider the displacement current in Ampère’s equation (2.10)

\[
\frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} \sim \frac{v^2 B_0}{c^2 a} \ll |\nabla \times \mathbf{B}| \sim \frac{B_0}{a},
\]

where we have assumed the length and time scales by \( a \) and \( t_0 \), respectively. These define a characteristic velocity \( v \sim a/t_0 \). Hence, the displacement vector has the order \( O(v^2/c^2) \) which is small compared to \( |\nabla \times \mathbf{B}| \), and therefore removed from Ampère’s law which now reads

\[
\mathbf{j} = \frac{1}{\mu_0} \nabla \times \mathbf{B}. \quad (2.17)
\]

\(^2\)Ideal MHD makes reference to the single fluid picture of the plasma.
2.2 Plasma physics description

The electric field term appearing in equation of motion (2.14) is estimated in the following way:

\[ \rho E \overset{\text{Eq. (2.11)}}{\sim} \epsilon_0 \frac{E^2}{a}, \]
\[ \overset{\text{Eq. (2.15)}}{\sim} \epsilon_0 \frac{v}{a} |\mathbf{v} \times \mathbf{B}|^2, \]
\[ \epsilon_0 = (\mu_0 c^2)^{-1} \frac{v^2 B^2}{\mu_0 a} \ll |\mathbf{j} \times \mathbf{B}| \sim \frac{B^2}{\mu_0 a}. \]  
(2.18)

Therefore, \( \rho E \) is of the order of \( O(v^2/c^2) \) which makes possible to neglect the space charge effects, and thus Eq. (2.11) is no longer needed. Finally, the gravitational contribution is completely negligible if we consider typical tokamak [34] parameters in which the following ratio holds

\[ \frac{|\mathbf{j} \times \mathbf{B}|}{|\mu g|} \ll 1. \]  
(2.19)

This means that the Lorentz force dominates gravitational effects for the phenomena of interest. Hence, we will neglect the term \( \rho g \) is the equation of motion.

For completeness, table 2.1 shows the quantitative criteria for the time scales needed to validate the ideal MHD model [35].

<table>
<thead>
<tr>
<th>Plasma physics time scales</th>
<th>Formulas</th>
<th>Numerical values (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Electron gyro period</td>
<td>( \tau_{ce} = \frac{2\pi}{\omega_{ce}} = \frac{2\pi m_e/q_e B_0}{2} )</td>
<td>( 7.1 \times 10^{-12} )</td>
</tr>
<tr>
<td>Electron plasma period</td>
<td>( \tau_{pe} = \frac{2\pi}{\omega_{pe}} = \frac{2\pi m_e c_0^2/nq_e^2}{2} )</td>
<td>( 7.9 \times 10^{-12} )</td>
</tr>
<tr>
<td>Ion plasma period</td>
<td>( \tau_{pi} = (m_i/m_e)^{1/2} \tau_{pe} )</td>
<td>( 4.8 \times 10^{-10} )</td>
</tr>
<tr>
<td>Ion gyro period</td>
<td>( \tau_{ci} = (m_i/m_e) \tau_{ce} )</td>
<td>( 2.6 \times 10^{-6} )</td>
</tr>
<tr>
<td>MHD time</td>
<td>( \tau_{MHD} = a/V_Ti )</td>
<td>( 2.3 \times 10^{-6} )</td>
</tr>
<tr>
<td>Electron-electron collision time</td>
<td>( \tau_{ce} = 7.4 \times 10^{-6} T_e^{3/2}/n )</td>
<td>( 1.0 \times 10^{-5} )</td>
</tr>
<tr>
<td>Ion-ion collision time</td>
<td>( \tau_{ii} = (2m_i/m_e)^{1/2} \tau_{ce} )</td>
<td>( 8.9 \times 10^{-4} )</td>
</tr>
<tr>
<td>Energy equilibration time</td>
<td>( \tau_{equil} = (m_i/2m_e) \tau_{ce} )</td>
<td>( 1.9 \times 10^{-2} )</td>
</tr>
<tr>
<td>Ignition time</td>
<td>( \tau_{ig} = 2.0/n )</td>
<td>1.0</td>
</tr>
<tr>
<td>Resistive diffusion time</td>
<td>( \tau_D = \mu_0 a^2/\eta )</td>
<td>( 1.4 \times 10^2 )</td>
</tr>
</tbody>
</table>

Table 2.1: Comparison of the characteristic MHD time with that of other basic plasma physics phenomena.

However, in this work it will be considered the resistive MHD. In particular in dissipative MHD the equilibrium profiles decay on a diffusion time \( \tau_D \) that is much longer than the characteristic Alfvén time \( \tau_A \) for ideal MHD (shown in
table). The typical instability $\lambda_r$ that we will study in resistive MHD, *lives* in
the following temporal window

$$\frac{1}{\tau_D} \ll \lambda_r \ll \frac{1}{\tau_A}. \quad (2.20)$$

Therefore, the instability should exponentiate much faster than the resistive
diffusion time, but much slower than the ideal MHD time. With the above
remarks, we are ready to go into the resistive magnetohydrodynamics approach
which is the developed in the following chapter.
Chapter 3

Physical and numerical models

3.1 Physical model

3.1.1 Resistive MHD

The starting point is the treaty of the resistive MHD equations. To present them, a modification of the ideal MHD model is done by introducing a resistive term in Ohm’s law. The set of dissipative MHD equations reads

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \quad (3.1)$$

$$\rho \left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \mathbf{v} = -\nabla p + \mathbf{j} \times \mathbf{B} \quad (3.2)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (3.3)$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} \quad (3.4)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (3.5)$$

where

$$\mathbf{E} + (\mathbf{v} \times \mathbf{B}) = \eta \mathbf{j} \quad (3.6)$$

where SI units are used. These relations represent the evolution equations for the mass density $\rho$, velocity of the fluid $\mathbf{v}$, magnetic field $\mathbf{B}$, and electric field $\mathbf{E}$; $c$ denotes the value of the velocity of light in vacuum while $\eta$ represents the resistivity of the magnetofluid. The Ohm’s law (3.6) determines the density current $\mathbf{j}(\mathbf{r}, t)$ whereas the free-divergence equation for the magnetic field (3.5) is used as an initial condition since for all time $t$ it is fulfilled $\partial (\nabla \cdot \mathbf{B}) / \partial t = -\nabla \cdot \partial \mathbf{B} / \partial t = \nabla \cdot (\nabla \times \mathbf{E}) = 0$ as the Faraday’s induction law demands.

In addition, a relation from gas dynamics for the evolution of the pressure $p(\mathbf{r}, t)$ is introduced

$$\frac{dp}{dt} + \gamma p \nabla \cdot \mathbf{v} \equiv \frac{\partial p}{\partial t} + \mathbf{v} \cdot \nabla p + \gamma p \nabla \cdot \mathbf{v} = 0 \quad (3.7)$$
where the total derivative in time is written as
\[
\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla
\]
which implies the use of the so-called Lagrangian time-derivative, evaluated while moving with the fluid, in contrast to the Eulerian time-derivative \( \frac{\partial}{\partial t} \), which is evaluated at a fixed position. Note that we have used the occasion to introduce the ratio of specific heats \( \gamma \equiv \frac{C_p}{C_V} \) as well.

However, it is possible to work out with other thermodynamic quantities such as the entropy per unit mass \( s \) or internal energy per unit mass \( u \) which allow us to replace \( p \) and \( \rho \), thus defining new basic variables. The former mentioned are defined by the ideal gas relations, with \( p = (n_e + n_i) k_B T \)

\[
\begin{align*}
  u & \equiv \frac{1}{\gamma - 1} \frac{p}{\rho} \approx C_V T, \quad C_V \approx \frac{(1 + Z) k_B}{(\gamma - 1) m_i} \\
  s & \equiv C_V \ln S + \text{const}, \quad S \equiv p \rho^{-\gamma}
\end{align*}
\]

here \( m_i \) is the mass of the ions, \( k_B \) is the Boltzmann constant, \( C_V \) and \( C_p \) are the specific heats at constant volume and pressure, respectively. From the heat conduction equation when both the thermal conduction and heat flow are neglected, i.e. an adiabatic process is taking place, the conduction equation reduces to

\[
\frac{dS}{dt} \equiv \frac{d}{dt} \left( \frac{p}{\rho^\gamma} \right) = 0
\]

\[
\Rightarrow \quad \frac{d}{dt} \left( \frac{p}{\rho^\gamma} \right) = \frac{\partial}{\partial t} \left( \frac{p}{\rho^\gamma} \right) + \mathbf{v} \cdot \nabla \left( \frac{p}{\rho^\gamma} \right) = 0
\]

this does not imply that the entropy is constant everywhere, but if the fluid is initially isentropic (has uniform entropy) then it will remain so. In a similar manner as the above formulation, a temporal evolution equation for the internal energy \( u \) could be obtained, nevertheless in our formulation the chosen state variable is \( S \).

In what follows, the derivation of the MHD equations with resistivity as a sole non-ideal effect and with sources neglected will be sketched. Direct substitution of Ampère law (3.4) into equation of motion (3.2) gives:

\[
\rho \left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \mathbf{v} = -\nabla p + \frac{1}{\mu_0} (\nabla \times \mathbf{B}) \times \mathbf{B}
\]

The equation of state (3.10) can be combined with the continuity equation (3.1):

\[
\frac{\partial p}{\partial t} = -\gamma p \nabla \cdot \mathbf{v} - \mathbf{v} \cdot \nabla p
\]
3.1 Physical model

Finally, applying the rotational operation on Ohm’s law (3.6) and introducing in the resulting equation both Eqs.(3.4) and (3.3) we obtain the following relation:

\[ \frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) - \frac{1}{\mu_0} \nabla \times (\eta \nabla \times \mathbf{B}) \]  

Eqs. (3.11), (3.12), and (3.13) form the full set of resistive MHD equations. As usual by means of a typical length, mass, and time scales the MHD equations can be made dimensionless [36, 37]. Thus, we normalize the radius to a plasma radius \( a \), the unit of time \( \tau \) follows from the relation between the plasma radius and the basic speed of macroscopic plasma dynamics, i.e. the Alfvén velocity

\[ V_A \equiv \frac{B_0(0)}{\sqrt{\mu_0 \rho_0}} \Rightarrow \tau \equiv \frac{a}{V_A} , \]

where the magnetic field \( B_0 \) and density \( \rho_0 \) are evaluated on axis. With the above mentioned, the transformation for the variables and differential operators reads

\[ \tilde{r} \equiv r/a, \quad \tilde{t} \equiv t/\tau, \]
\[ \tilde{\nabla} \equiv a \nabla, \quad \partial/\partial \tilde{t} \equiv \tau \partial/\partial t, \]
\[ \tilde{\mathbf{v}} \equiv \mathbf{v}/V_A, \quad \tilde{\mathbf{B}} \equiv \mathbf{B}/B_0(0), \]
\[ \tilde{\rho} \equiv \rho/\rho(0), \]
\[ \tilde{\eta} \equiv \eta/(\mu_0 a V_A) \]
\[ \tilde{\mathbf{j}} \equiv \mu_0 a/B_0(0) \mathbf{j} . \]  

Thus, an important result is shown: the resistive MHD equations are scale independent, i.e. they do not depend on the size of the plasma \( a \), the magnitude of the magnetic field \( B_0(0) \) and on the density \( \rho_0(0) \). For obvious reasons the tilde symbol is no longer used, thus the variables appearing in Eqs. (3.15)-(3.17) will remain with that notation in this work.

3.1.2 Equilibrium, stability and linearization analysis

Since the MHD approach describes how magnetic, intertial, and pressure forces interact within a resistive (or ideal perfectly conducting) plasma in an given
geometry, it is now the turn of the MHD equilibrium and stability analysis
that could lead to the discovery of attractive magnetic geometries for future
fusion reactors. The latter suggests to develop a theoretical approach to plasma
confinement for a given magnetic geometry. Typically, the path for developing
such a study consists of three stages:

1. Fixing an equilibrium state (which should be pertinent for the case
study).

2. Followed by the determination of the types of waves produced by per-
turning this state.

3. Examining whether those perturbations lead to instabilities that would
destroy the actual configuration.

Once the equilibrium condition is established, which typically involves an
additional flowless assumption, the equilibrium is subjected to small\(^1\) per-
turbations. This involves the study of linearized and time-dependent MHD
equations which is the main topic of this section.

The background equilibrium is chosen to be a static one, i.e. a flowless
condition is assumed \( v \equiv v_0 = 0 \), from the Eq. of motion together with
Ampère’s and the Gauss law we have

\[
\mathbf{j}_0 \times \mathbf{B}_0 = \nabla p_0, \quad \mathbf{j}_0 = \nabla \times \mathbf{B}_0, \quad \nabla \cdot \mathbf{B}_0 = 0 \tag{3.18}
\]

the appropriate boundary conditions (b.c.s) are needed as well. With these
expressions together with an initial choice of one of the profiles, it is possible
to determine the whole equilibrium variables \( \rho_0 (\mathbf{r}), p_0 (\mathbf{r}), \mathbf{j}_0 (\mathbf{r}), \) and \( \mathbf{B}_0 (\mathbf{r}) \). We
are now ready to introduce the perturbation from the equilibrium in the follo-
wing manner:

\[
\begin{align*}
\mathbf{v} (\mathbf{r}, t) &= \mathbf{v}_1 (\mathbf{r}, t) , \\
p (\mathbf{r}, t) &= p_0 (\mathbf{r}) + p_1 (\mathbf{r}, t) , \\
\mathbf{B} (\mathbf{r}, t) &= \mathbf{B}_0 (\mathbf{r}) + \mathbf{b}_1 (\mathbf{r}, t) .
\end{align*}
\tag{3.19}
\]

where \( p_0 \) and \( \mathbf{B}_0 \) corresponds to an homogeneous equilibrium, satisfying Eqs.
(3.18) and the b.c.s. The time dependence enters in the perturbed variables
\( f_1 (\mathbf{r}, t) \) and its explicit dependence on the \( \mathbf{r} \) and \( t \) variables is determined be-
low. Furthermore, the perturbations \( f_1 (\mathbf{r}, t) \) should satisfy \( |f_1 (\mathbf{r}, t)| \ll |f_0 (\mathbf{r})| \)
(this condition does not apply to the velocity \( \mathbf{v} \)). The derived first order

\(^1\)Small enough in order to considered these perturbations in the linear regime.
3.1 Physical model

The equations for the perturbations of Eqs. (3.15)-(3.17) are:

$$\rho_0 \frac{\partial \mathbf{v}_1}{\partial t} = -\nabla p_1 + (\nabla \times \mathbf{B}_0) \times \mathbf{b}_1 + (\nabla \times \mathbf{b}_1) \times \mathbf{B}_0, \quad (3.20)$$

$$\frac{\partial p_1}{\partial t} = -\mathbf{v}_1 \cdot \nabla p_0 - \gamma p_0 \nabla \cdot \mathbf{v}_1, \quad (3.21)$$

$$\frac{\partial \mathbf{b}_1}{\partial t} = \nabla \times (\mathbf{v}_1 \times \mathbf{B}_0) - \nabla \times (\eta_0 \nabla \times \mathbf{b}_1). \quad (3.22)$$

Here, the equilibrium quantities $\cdot_0$ are assumed to be known from the unperturbed system of equations and the perturbed quantities $\cdot_1$ are the unknown variables to be determined.

Generally speaking the stability study can be conducted by means of two methods, namely the so-called energy method and by solving the coupled partial differential equations directly. The former involves a variational formulation of the problem in which a plasma displacement vector field $\xi(\mathbf{r},t)$ is introduced and then a potential energy functional $W[\xi]$ together with a kinetic energy functional $K[\dot{\xi}]$ determine the stability of the system. On the other hand, the second method relies on the elegant force operator formalism $\mathbf{F}(\xi)$ leading to an equation of motion for the plasma displacement $\xi$ that it is then solved; a clear and pedagogical development of such approach is found in [38, 39, 40].

However, since this work is dedicated to develop a resistive MHD code, we are forced to consider the seven components of $\{\mathbf{v}_1, p_1, \mathbf{b}_1\}$ describing the perturbed state of the system. This is needed because the resistive term spoils the possibility of integrating the above equations directly, which eventually lead to expressions in terms of the displacement vector $\xi$ (the ideal MHD formerly mentioned). In the resistive MHD problem one typically leads with non-Hermitian operators and to my concern there is no general unique technique to deal with it. To do such analysis, the present work focus on the solution via the finite element method (FEM) together with the Galerkin scheme; both will be the central topic of Section 3.2. But before I delve into the numerical approach, an explicit geometry of our problem is established, i.e. the cylindrical plasma model is presented, which is one of the most widely studied model in plasma stability theory.

3.1.3 Cylindrical plasmas

The diffuse cylindrical plasma column of radius $a$ and length $L$ with a helical magnetic field $\mathbf{B}$ (Fig. 3.1) is our starting point.
Assuming a magnetic field profile $B_0 = B_\theta(r)e_\theta + B_z(r)e_z$, in cylindrical $r$, $\theta$, and $z$-coordinates, the equilibrium equations (3.18),
\[
\begin{align*}
\mathbf{j}_0 \times \mathbf{B}_0 &= \nabla p_0, \\
\mathbf{j}_0 &= \nabla \times \mathbf{B}_0, \\
\nabla \cdot \mathbf{B}_0 &= 0,
\end{align*}
\]
reduce to
\[
\frac{dp_0}{dr} = j_\theta B_z - j_z B_\theta, \\
\left[ p(r) + \frac{1}{2} B_\theta^2(r) \right]' + \frac{B_\theta^2(r)}{r} = 0.
\]
(3.24)

Therefore, with two profiles given, Eq.(3.24) can be solved to obtain the remaining one. This fact will be used once a pair of profiles are explicitly given. In addition, since a periodic cylinder representation of tokamaks will be used, for most axisymmetric toroidal configurations typically the safety factor $q(r)$ is introduced
\[
q(r) = \frac{r}{R_0} \frac{B_z(r)}{B_\theta(r)},
\]
(3.25)
where $R_0$ is the major toroidal radius which is related to the cylinder’s length by $2\pi R_0 = L$ as seen in Fig. 3.2.
3.1 Physical model

The safety factor has a crucial role in plasma MHD stability studies by means of the Suydam’s criterion [41] which tells us the necessary condition for stability of the so-called interchange modes [42, 43, 44]

\[
p'(r) + \frac{1}{8} r B_z^2 \left( \frac{q'(r)}{q(r)} \right)^2 > 0. \tag{3.26}
\]

Its violation gives rise to highly localized instabilities driven by pressure gradients \(p'(r)\) which interchange the magnetic field lines without appreciable bending. A concrete application of this criterion will be discussed later on.

We come back to the system of linearized MHD equations

\[
\rho_0 \frac{\partial \mathbf{v}_1}{\partial t} = -\nabla p_1 + (\nabla \times \mathbf{B}_0) \times \mathbf{b}_1 + (\nabla \times \mathbf{b}_1) \times \mathbf{B}_0, \tag{3.27}
\]

\[
\frac{\partial p_1}{\partial t} = -\mathbf{v}_1 \cdot \nabla p_0 - \gamma p_0 \nabla \cdot \mathbf{v}_1, \tag{3.28}
\]

\[
\frac{\partial \mathbf{b}_1}{\partial t} = \nabla \times (\mathbf{v}_1 \times \mathbf{B}_0) - \nabla \times (\eta_0 \nabla \times \mathbf{b}_1), \tag{3.29}
\]

\[
\nabla \cdot \mathbf{b}_1 = 0. \tag{3.30}
\]

Here, the last equation serves merely as an initial condition. It is invoked the normal mode expansion by exploiting both the rotational symmetry in \(\theta\) and the traslational symmetry in \(z\). Thus, the ansatz for the perturbed quantites has the form

\[
f_1(r, \theta, z, t) = f(r) e^{i(m\theta+nkz+\omega t)}, \tag{3.31}
\]

where \(f(r)\) is the amplitude of our perturbation, \(\omega\) the frequency, \(m\) and \(n\) are the poloidal and toroidal mode numbers, respectively. Furthermore, since we are interested in the straight tokamak limit a periodicity length is defined via

**Figure 3.2**: ‘Straight tokamak’ limit. Periodic cylinder representation of tokamaks with length \(2\pi R_0\).
3. Physical and numerical models

The selected Fourier expansion transforms our original system of partial differential equations (PDEs) into an eigenvalue problem\(^3\)

\[
i \omega \rho_0 \frac{v_1}{r} = - \left[ \frac{\dot{p}}{r} + \frac{1}{m} B_\theta b_1' + \left( B_z - \frac{n k r}{m} B_\theta \right) b_3' \right] + \frac{1}{r} \left( \frac{m}{r} B_\theta + n k B_z \right) b_1
\]

\[
- \frac{2}{r m} B_\theta b_1' + \frac{2 n k}{r m} B_\theta b_3,
\]

\[
i \omega \rho_0 r v_2 = \frac{m}{r} \ddot{p} + \left( \frac{1}{r} B_\theta + B_\theta' \right) b_1 - \frac{n k r}{m} B_\theta b_1' + \left( \frac{n^2 k^2 r}{m} + \frac{m}{r} \right) B_z b_3,
\]

\[
i \omega \rho_0 \frac{v_3}{r} = \frac{n k}{r} \ddot{p} - \left( \frac{m}{r^2} + \frac{n^2 k^2}{m} \right) B_\theta b_3 + \frac{n k}{m} B_\theta b_1' + B_z \frac{1}{r} b_1,
\]

(3.32)

\[
i \omega \rho_0 \frac{\dot{v}_3}{r} = - \frac{1}{r} p_0' v_1 - \gamma p_0 \frac{1}{r} v_1' - \gamma p_0 \frac{m}{r} v_2 - \gamma p_0 \frac{n k}{r} v_3,
\]

\[
i \omega b_1 = - \left( \frac{m}{r} B_\theta + n k B_z \right) v_1 + \eta_0 \left[ b_1'' + \frac{1}{r} b_1' - \left( \frac{m^2}{r^2} + n^2 k^2 \right) b_1 - \frac{2 n k}{r} b_3 \right],
\]

\[
i \omega b_3 = - B_z v_1' - m B_z v_2 + \frac{m}{r} B_\theta v_3 - B_\theta' v_1 + \eta_0 \left[ \left( b_3' - \frac{b_3}{r} \right)' - \left( \frac{m^2}{r^2} + n^2 k^2 \right) b_3 \right].
\]

The change of variables

\[
v_1 = rv_r, \quad v_2 = iv_\theta, \quad v_3 = i rv_z,
\]

\[
\dot{p} = rp_1, \quad b_1 = i r b_r, \quad b_3 = r b_z,
\]

(3.33)

is done to preserve the 6 components \(\{v_1, v_2, v_3, \dot{p}, b_1, b_3\}\) as real quantities [37]. The poloidal component \(b_3 \equiv b_2\) of the perturbed \(b\)-field is eliminated by means of the divergence-free condition

\[
\nabla \cdot b_1 \overset{(A.34), \text{anzatz}}{=} - \frac{i}{r} (b_1' - m b_2 - n k b_3) = 0,
\]

(3.34)

which will be valid for poloidal modes \(m \neq 0\). If the poloidal mode \(m = 0\) is to be studied, then \(b_3\) should be eliminated provided \(n k \neq 0\). The above eigenvalue problem of Eq.(3.32) consists of a system of coupled ordinary differential equations (ODEs) which can be symbolically represented as

\[
i \omega \tilde{u} = \mathbb{D} u,
\]

where we have introduced a six-dimensional state vector \(u\)

\[
u^T = (v_1, v_2, v_3, \dot{p}, b_1, b_3).
\]

\(^3\)After a tedious and lengthy work.
The spatial differential operators and equilibrium quantities are contained in the operators $I$ and $D$. For instance, due to the definition of the state vector $u$ as in Eq. (3.35), the $I$ operator has a diagonal form. However, the problem is not yet well-defined since a unique solution is found only if the appropriate b.c.s are taken into account. Therefore, in the next section we are going to present two case studies for the plasma interface that we are interested in, namely the plasma-wall and plasma-vacuum-wall. We bear in mind that the goal is to derive expressions that will be used in the numerical solution of the eigenvalue problem (3.32).

Plasma interfaces

Laboratory plasmas typically are surrounded by a vacuum vessel together with a wall which separates the plasma and the outer components of the tokamak. For this reason, a physical understanding and a mathematical development of the b.c.s associated to plasma-wall and plasma-vacuum-wall interfaces need to be established.

Ideal boundary conditions

1. Plasma-wall interface This is the case of the cylindrical and perfectly conducting plasma confined inside a rigid wall, i.e. the plasma is isolated from the outside world by a wall, which is a perfect conductor, located at the plasma radius $a$, see Fig. 3.3. At the wall, both the normal magnetic field and the normal component of the velocity vanish

$$n \cdot B |_{r=a} = 0 \quad (\text{at the wall}),$$

$$n \cdot v |_{r=a} = 0 \quad (\text{at the wall}).$$

These b.c.s guarantee the conservation of the mass, momentum, energy, and magnetic flux, so that the system is closed.

2. Plasma-vacuum-wall interface In this situation the plasma is confined inside a perfectly conducting wall and isolate from it by a vacuum region\(^4\). On the one hand, the plasma-vacuum interface is located at $r = a$ whereas the vacuum-wall interface is found at $r = b$, see Fig. 3.4. The dynamics of the electric and magnetic vacuum fields $\hat{E}$ and $\hat{B}$ follow the Maxwell’s equations in their non-relativistic form

$$\nabla \times \hat{B} = 0, \quad \nabla \cdot \hat{B} = 0,$$

$$\nabla \times \hat{E} = -\frac{\partial \hat{B}}{\partial t}, \quad \nabla \cdot \hat{E} = 0.$$  

\(^4\)By vacuum region we mean a region of low enough density -compared to the plasma one- to be treated as a vacuum.
Regarding the vacuum-wall interface, the normal component of the vacuum magnetic field \( \hat{\mathbf{B}} \) vanish at the conducting wall

\[
\mathbf{n} \cdot \hat{\mathbf{B}}_{r=b} = 0 \quad (\text{at the conducting wall}).
\] (3.40)

Now, if we concentrate on the plasma-vacuum interface, the proper boundary conditions read

\[
\mathbf{n} \cdot \mathbf{B}_{r=a} = \mathbf{n} \cdot \hat{\mathbf{B}}_{r=a} = 0 \quad (\text{at plasma-vacuum interface}),
\] (3.41)
\[
\left( p + \frac{B^2}{2} \right)_{r=a} = \frac{1}{2} \hat{B}^2_{r=a} \quad (\text{at plasma-vacuum interface}),
\] (3.42)

where Eq. (3.42) expresses the total pressure balance and Eq. (3.41) reflects the fact that the field lines cannot point out of the plasma which avoids the plasma to freely flow along the magnetic field lines into the vacuum region.

Finally, all quantities at the origin \( r = 0 \) are assumed to be known,

\[
v_1(0) = 0, \quad v_3(0) = 0, \quad \dot{p}(0) = 0,
\] (3.43)
\[
b_1(0) = 0, \quad b_3(0) = 0.
\] (3.44)

With these interfaces and their associated b.c.s, we have fully developed the physical model. But now we will come to the point that was mentioned a couple of times already: the numerical solution of the eigenvalue problem (EVP)

\[
i \omega \mathbb{I} \mathbf{u} = \mathbb{D} \mathbf{u},
\] (3.45)

where \( \mathbf{u}^T = (v_1, v_2, v_3, \dot{p}, b_1, b_3) \), \( \mathbb{I} \) and \( \mathbb{D} \) are matrices with differential operators and equilibrium quantities explicitly given in the system of equations (3.32).
3.2 Numerical model

3.2.1 Finite element method: Galerkin scheme

In this section the finite element method (FEM), viz. the Galerkin scheme and its weak formulation, will be illustrated. Since our system of ODEs (3.32) deals with at most second-order derivatives, it is reasonable to start our treatise with the general 1D second-order differential equation of the form

\[-\frac{d}{dx} \left[p(x)\frac{du(x)}{dx}\right] + q(x)u(x) = f(x),\]

(3.46)

defined on the finite closed interval \([0, 1]\) where the coefficient functions \(p(x)\) and \(q(x)\) and the inhomogeneous term \(f(x)\) are given and real valued. We are considering the self-adjoint form of the ODE (3.46) since a similar structure appears in the system of differential equations (3.32). On the other hand, the solution of our problem \(u(x)\) will be subjected to a Dirichlet-type boundary condition at \(x = 0\)

\[u(x)\bigg|_{x=0} = 0,\]

(3.47)

whereas at \(x = 1\) the solution \(u(x)\) will present an inhomogeneous mixed boundary condition, i.e.

\[\left[\alpha u(x) + \beta \frac{du(x)}{dx}\right]_{x=1} = F,\]

(3.48)

here \(\alpha, \beta\) and \(F\) are constants.

The numerical approximation of our one dimensional ODE Eq. (3.46) will be performed in the framework of the FEM. Briefly, the FEM can be described as a three-step method, namely
1. Definition of a mesh that discretizes the space domain.

2. The exact solution of the ODE (or PDE) is approximated by linear combination of local piecewise polynomials.

3. Selection of suitable coefficients in the linear combination such that the error of the approximation is minimized.

We now proceed to the description of the these steps. The first one consists on dividing the interval of interest in sub-domains \([x_{i-1}, x_i]\) for \(i = 1, \ldots, N\). These sub-domains can be either uniform or non-uniform spaced and they will be referred as the ‘elements’ connecting the ‘nodes’ \(x_i\). The solution \(u(x)\) is then approximated by a linear combination of basis functions \(h_i(x)\)

\[
u(x) \approx \tilde{u}(x) = \sum_{i=0}^{N} a_i h_i(x), \tag{3.49}
\]

where the basis are locally piecewise polynomials\(^5\) that are defined in the sub-intervals \([x_{i-1}, x_{i+1}]\) and vanish outside these ones, i.e.

\[
h_i(x) = \begin{cases} 
P(x; x_{i-1}, x_i) & \text{for } x_{i-1} \leq x \leq x_i, \\
Q(x; x_i, x_{i+1}) & \text{for } x_i \leq x \leq x_{i+1}, \\
0 & \text{elsewhere.}
\end{cases} \tag{3.50}
\]

Moreover, approximations to derivatives are obtained by differentiating Eq. (3.50). The schematic representation of the above mentioned procedure is shown in figure 3.5.

Since \(\tilde{u}(x)\) only approximates the real solution \(u(x)\), we foresee that introducing \(\tilde{u}(x)\) in the differential equation in its self-adjoint form (3.46) we will not obtain zero but a residual function instead that will depend on the form of \(\tilde{u}(x)\):

\[
R_{\tilde{u}} \left( x; \{a_i\}_{i=0}^{N} \right) \equiv \sum_{i=0}^{N} a_i \left[ -\frac{d}{dx} \left( p(x) \frac{d\tilde{u}(x)}{dx} \right) + q(x) \tilde{u}(x) - f(x) \right] \neq 0,
\]

\[
= \sum_{i=0}^{N} a_i \left[ -\frac{d}{dx} \left( p(x) \frac{dh_i(x)}{dx} \right) + q(x) h_i(x) \right] - f(x) \neq 0.
\]

\[
= \sum_{i=0}^{N} a_i \left[ -\frac{d}{dx} \left( p(x) \frac{dh_i(x)}{dx} \right) + q(x) h_i(x) \right] - f(x) \neq 0. \tag{3.51}
\]

Now the idea is to decide on some method to determine the coefficients \(a_i\) such that the residual function \(R_{\tilde{u}} \left( x; \{a_i\}_{i=0}^{N} \right)\) becomes close to zero over a chosen domain (this would mean that our approximation gets closer to the exact

\(^5\)The polynomials are constructed from element shape functions of a local coordinate \(\xi\), typically defined in the interval of the local coordinate \(\xi \in [-1, 1]\). The piecewise shape functions can be linear, quadratic and/or cubic and of course, they will be conveniently chosen from problem to problem.
3.2 Numerical model

Figure 3.5: Schematic procedure of the first two steps of the FEM: space discretization and approximation to the exact solution of the differential equation.

One possibility is to force the residual to zero in some average sense over the domain \( \Omega \). That is

\[
\int_{\Omega} w_j(x) R_\tilde{u} (x; a_0, \ldots, a_N) \, dx = 0,
\]

where the number of weight functions \( w_j(x) \) is exactly equal to the number of unknown constants \( a_i \) in \( \tilde{u}(x) \). The result is a set of \( N+1 \) algebraic equations for the unknown constants \( a_i \) which in principle can be solved (once the weight function \( w_j(x) \) is selected). The form of (3.52) is known as the *weighted residual formulation* and it can be found in different flavors regarding the particular choice of weight functions. When the weight functions \( w_j(x) \) are chosen equal to the basis functions \( h_j(x) \), i.e. \( w_j = h_j \forall j \in \{0, \ldots, N\} \), the residual fulfills the following condition

\[
\int_0^1 h_j(x) R_\tilde{u} (x; a_0, \ldots, a_N) \, dx = 0 \quad (3.51)
\]

Thus, the residual is made orthogonal to the function space spanned by the basis functions, in the sense of the inner product defined by the above integral over the domain \([0, 1]\). This special case of the weight residual formulation is called *Galerkin method* and it will be the core of our numerical approach.

So far we have just illustrated that the Galerkin scheme can be established in 3 steps: first, it is performed a domain discretization followed by an approximation of the ODE’s solution in terms of some basis functions. Next, the error in the approximation is minimized through the determination of a weight
function which will force the residual to zero. Finally, a linear algebraic system of $N + 1$ equations for the $N + 1$ unknowns $\{a_j\}_{j=0,1,...,N}$ pops out and then solved. For completeness, figure 3.6 summarizes the basic ideas behind the Galerkin method.

\[ u(x), \tilde{u}(x) \]

\[ u(x) = \tilde{u}(x) = \sum_{i=0}^{N} a_i h_i(x) \]

\[ \int_{0}^{1} h_j \left[ \left( -\frac{d}{dx} p(x) \frac{d}{dx} + q(x) \right) \tilde{u}(x) - f(x) \right] dx = 0, \quad j = 0, 1, ..., N, \quad (3.54) \]

Despite that we have developed most of the machinery for our numerical model, it is common to simplify the integrals appearing in the Galerkin method. Thereby, reducing the degree of derivatives of our problem and thus admitting wider variety of possible solution functions. This is the so-called weak formulation of the Galerkin method and it will be described in the following section.

3.2.2 A weak formulation

The basic idea behind weak formulation of the Galerkin scheme is that the residual’s orthogonality condition, viz. Eq. (3.53), suffers a slight modification by performing integration by parts. This lowers the degree of the involved derivatives and imposes less restrictions on the smoothness of the solution function $\tilde{u}(x)$. To achieve this weak form, we rewrite the residual’s relation as
where we have used the definition of $\tilde{u}(x)$ by Eq. (3.49) for convenience. Now we perform the mentioned integration by parts on the integrands with the highest order derivatives

$$\int_0^1 h_j \left[ -\frac{d}{dx} \left( p(x) \frac{d\tilde{u}(x)}{dx} \right) + q(x)\tilde{u}(x) - f(x) \right] dx =$$

$$- \left[ h_j p(x) \frac{d\tilde{u}}{dx} \right]_0^1 + \int_0^1 \frac{dh_j}{dx} p(x) \frac{d\tilde{u}}{dx} dx + \int_0^1 h_j(x) [q(x)\tilde{u}(x) - f(x)] dx ,$$

and recalling Eq. (3.54), we know that $\forall j = 0, \ldots, N$, the following relation holds

$$- \left[ h_j p(x) \frac{d\tilde{u}}{dx} \right]_0^1 + \int_0^1 \frac{dh_j}{dx} p(x) \frac{d\tilde{u}}{dx} dx + \int_0^1 h_j(x) [q(x)\tilde{u}(x) - f(x)] dx = 0 .$$

(3.55)

Equation (3.55) is the final form of the Galerkin scheme (weak formulation) that will be implemented on our MHD problem. When working in this form only the first-order derivative of $\tilde{u}(x)$ is required, which in turn relaxes the regularity condition of the solution function. The advantage of the weak form is that functions with discontinuous derivatives are now allowed, whereas in the original differential equation (3.46) the first order derivatives have to fulfill both continuity and differentiability conditions.

As a final remark, the actual integration by parts already gives the needed boundary terms for implementing the actual b.c.s. For example, in our initial 1D model (3.46) this yields for the current b.c.s:

$$- \left[ h_j p(x) \frac{d\tilde{u}}{dx} \right]_0^1 \text{ b.c.s (3.47), (3.48)} = \delta_j N h_N(1) p(1) \left( \frac{\alpha}{\beta} h_N(1) a_N - \frac{F}{\beta} \right) ,$$

$$j = 0, 1, \ldots N ,$$

$h_N$ is the only basis function different from zero in $x = x_N = 1$ from its definition in Eq. (3.50), whereas the $h_0$ should be dropped to satisfy boundary condition (3.47), i.e. all basis functions that are non-zero at the origin have to be left out . The former b.c. is called natural boundary condition since it can enter the equation in a direct way and the latter b.c. that has to be imposed explicitly is called essential boundary condition. Moreover, it is this boundary term that will allow us to implement the plasma interface cases illustrated in Section 3.1.3.
3.2.3 MHD EVP in the weak form

Now we bring into context the numerical model formerly described. Recalling the actual one dimensional MHD problem and its system of ODEs (3.32)

\[
\begin{align*}
\iota \omega \rho_0 \frac{v_1}{r} &= - \left( \hat{\rho} + \frac{1}{m} B_\theta b_1' + \left( B_z - \frac{nkr}{m} B_\theta \right) b_3' \right) + \frac{1}{r} \left( \frac{m}{r} B_\theta + nk B_z \right) b_1 \\
&\quad - \frac{2}{rm} B_\theta b_1' + \frac{2nk}{rm} B_\theta b_3, \\
\iota \omega \rho_0 r v_2 &= \frac{m}{r} \hat{\rho} + \left( \frac{1}{r} B_\theta + B_\theta' \right) b_1 - \frac{nk}{m} B_z b_3' + \left( \frac{n^2k^2 r}{m} + \frac{m}{r} \right) B_z b_3, \\
\iota \omega \rho_0 \frac{v_3}{r} &= \frac{nk}{r} \hat{\rho} - \left( \frac{m}{r^2} + \frac{n^2k^2}{m} \right) B_\theta b_3 + \frac{nk}{m} B_\theta b_3' + \frac{1}{r} B_z b_1, \\
\iota \omega \frac{\hat{\rho}}{r} &= - \frac{1}{r} p_0' v_1 - \gamma p_0 \frac{1}{r} v_1' + \frac{m}{r} v_2 - \gamma p_0 \frac{nk}{r} v_3, \\
\iota \omega b_1 &= - \left( \frac{m}{r} B_\theta + nk B_z \right) v_1 + \eta_0 \left[ b_1'' + \frac{1}{r} b_1' - \left( \frac{m^2}{r^2} + n^2k^2 \right) b_1 - \frac{2nk}{r} b_3 \right], \\
\iota \omega b_3 &= - B_z v_1' - m B_z v_2 + \frac{m}{r} B_\theta v_3 - B'_z v_1 + \eta_0 \left[ b_3' - \frac{b_3}{r} \right] - \left( \frac{m^2}{r^2} + n^2k^2 \right) b_3.
\end{align*}
\]

this system of equations may be written in matrix form as

\[
\mathbb{D} \mathbf{u} = \iota \omega \mathbb{I} \mathbf{u} ,
\]

where \( \mathbf{u}^T = (v_1, v_2, v_3, \hat{\rho}, b_1, b_3) \equiv (u^1, u^2, u^3, u^4, u^5, u^6) \) is the state vector (where superscript associates each component of the vector with a unique state quantity), \( \mathbb{I} \) and \( \mathbb{D} \) are matrices with differential operators and equilibrium quantities as before. The weak formulation is now applied to the above equation, requesting that the \( k \)-th component of the state vector \( \mathbf{u}(r) \) is approximated by

\[
u^k(r) \approx \tilde{u}^k(r) = \sum_{i=0}^{N} a^k_i h^k_i(r) \quad k = 1, 2, \ldots, 6,
\]

same as before, \( h^k_i(r) \) are the chosen expansion functions and \( a^k_i \) are the coefficients to be determined. Multiplying Eq. (3.56) by an element of the chosen
Numerical model

basis, say $h_j$, and then integrating over the domain of interest $\Omega^6$, we have:

$$\int_\Omega (D\hat{u} - i\omega \mathbb{I}\hat{u}) h_j \, dr = 0,$$

$$\langle L\hat{u} | h_j \rangle = 0,$$

(3.58)

where we have introduced the operator $L = D - i\omega \mathbb{I}$ and Dirac’s notation of inner product only for showing that above integral satisfies the condition (3.54). After application of the weak formulation (3.55) on our resistive MHD problem (3.58) a complex EVP for the coefficients $a^k$ should be solved

$$Aa = i\omega Ba,$$

(3.59)

with $a$ the vector of the $6N$ expansions coefficients, the matrices $A$ and $B$ consist of $N \times N$ subblocks and dimension $n \times n$, where $n$ is the number of linear independent basis elements $h_j$ of the expansion.

Now, the matrix subblocks $A_{ji}$ that are generated after application of Galerkin scheme in its weak formulation will be explicitly given. We start as follows, assume that we want to calculate the matrix element $A_{ji}$ associated to the first linearized MHD relation of our system of ODEs

$$i\omega \rho_0 \frac{v_1}{r} = -\left[\frac{\dot{p}}{r} + \frac{1}{m}B_b b_1' + \left(B_z - \frac{nkr}{m} B_\phi \right) \frac{b_2}{r} \right]' + \frac{1}{r} \left(\frac{m}{r} B_\theta + nk B_z\right) b_1$$

$$- \frac{2}{rm} B_b b_1' + \frac{2nk}{rm} B_\theta b_3.$$

Then, the weight function $h_j$, which appears in $\langle D\hat{u} - i\omega \mathbb{I}\hat{u} | h_j \rangle$, that corresponds to the expansion of radial velocity $v_1$ will only interact with the above equation when doing the inner product. Next, for the diagonal part $i\omega \mathbb{I}$, which solely the state variable $v_1$ is involved, a matrix subblock is generated and only contains the interaction between the basis expansions of the radial velocity, i.e. $h^1 - h^1$. On the other hand, for the non-diagonal part, $D$, other state variables appear, namely $\dot{p}$, $b_1$, and $b_3$, thus other matrix subblocks are generated regarding the interaction between the weight function $h^1$ and the triad expansion functions $h^4, h^5, \text{ and } h^6$ (associated to $\dot{p}, b_1, \text{ and } b_3$, respectively).

---

6In the case of the plasma surrounded by a wall, the domain $\Omega$ is defined by the unit interval $[0, 1]$, whereas in the free-boundary mode case, plasma-wall distance $r_{b/a} \equiv b/a$ should be taken into account, i.e. the domain of interest is $[0, r_{b/a}]$, where $r_{b/a}$ means the relative position of the wall respect the plasma.
Mathematically, the latter reads as:

\[ \sum_{i=0}^{N} a_i^1 \int \frac{h_i^1}{r} \rho_0 h_j^1 dr = - \sum_{i=0}^{N} \int \left[ \frac{a_i^4 h_i^4}{r} + \frac{1}{m} B_\theta a_i^3 h_i^3' \right]' h_j^1 dr \]

\[ - \sum_{i=0}^{N} \int \left[ \left( B_z - \frac{nkr}{m} B_\theta \right) a_i^6 h_i^6' \right]' h_j^1 dr \]

\[ + \sum_{i=0}^{N} \int \frac{a_i^4 h_i^4}{r} \left( \frac{m}{r} B_\theta + nk B_z \right) h_j^1 dr \]

\[ - \sum_{i=0}^{N} \int \left[ a_i^4 h_i^4 \frac{2}{mr} B_\theta + a_i^6 h_i^6' \frac{2nk}{mr} B_\theta \right] h_j^1 dr. \]

Rearranging a bit, we can write the above equations as:

\[ i \omega \sum_{i=0}^{N} a_i^1 B_{ji} (1, 1) = \sum_{i=0}^{N} \left[ a_i^4 A_{ji} (1, 4) + a_i^5 A_{ji} (1, 5) + a_i^6 A_{ji} (1, 6) \right], \]

where a subblock \( A_{ji} (\cdot, \cdot) \) of the matrix \( A \) is generated by pairing the \( i \)-th and \( j \)-th elements of the approximation (3.57), whereas the numbers appearing inside brackets stand for the specific basis elements that are selected. As before, the prime symbol denotes the derivative with respect to the \( r \) variable, it is implicitly assumed the \( r \) dependence of the basis expansion functions \( h_i \), i.e. \( h_i = h_i(r) \), and the domain of integration \( \Omega \), its boundary \( \partial \Omega \) together with the b.c.s will depend on the case study under consideration, viz. plasma-wall or plasma-vacuum-wall for both ideal and resistive MHD problem. It is worth noticing that the above relation resembles the matricial form \( A a = i \omega B a \) of Eq. (3.59).

In the same fashion, the remaining relations of our system of ODEs are treated. For instance, in the second relation only the basis function associated to the expansion of the variable \( \psi_2 \) will act as the weight function, whereas the number of matrix subblocks that are generated will depend upon the number of different state variables inside this particular equation, and so on.

**A and B subblocks**

The non-diagonal matrix subblocks \( A_{ji} (1, \beta) \) \( \forall \beta = 4, 5, 6 \) that are related to our first MHD equation reads

\[ A_{ji} (1, 4) = \int_{\Omega} \left[ -\frac{1}{r} h_i^4 \right]' h_j^1 dr \text{ \quad \text{integration by parts}} \]

\[ = - \left[ h_i^1 h_i^4 \right]_{\partial \Omega} + \int_{\Omega} h_i^4 \frac{1}{r} h_j^1 dr, \]

\[ (3.60) \]
\[ A_{ji}(1,5) = - \int_{\Omega} dr \left( \frac{B_\theta}{m} \hat{h}_5^i \right) h_j^1 - \int_{\Omega} dr \frac{B_\theta}{m} \hat{h}_5^i h_j^1 + \int_{\Omega} dr \frac{h_5^i}{r} \left[ \frac{m}{r} B_\theta + nkB_z \right] h_j^1 + \int_{\Omega} dr \hat{h}_5^i \left[ -\frac{2}{mr} B_\theta \right] h_j^1 \]

\[ = - \left[ h_5^i \frac{B_\theta}{m} h_j^i \right]_{\partial \Omega} + \int_{\Omega} dr \frac{h_5^i}{r} \left[ \frac{m}{r} B_\theta + nkB_z \right] h_j^1 + \int_{\Omega} dr \hat{h}_5^i \left[ -\frac{2}{mr} B_\theta \right] h_j^1, \quad (3.61) \]

\[ A_{ji}(1,6) = - \int_{\Omega} dr \left[ (B_z - \frac{nkr}{m} B_\theta) \frac{h_6^i}{r} \right] h_j^1 + \int_{\Omega} dr \hat{h}_6^i (\frac{2nk}{mr} B_\theta h_j^1) + \int_{\Omega} dr \hat{h}_6^i \left[ -\frac{2}{mr} B_\theta \right] h_j^1 \]

\[ = - \left[ \frac{h_6^i}{r} (B_z - \frac{nkr}{m} B_\theta) h_j^1 \right]_{\partial \Omega} + \int_{\Omega} dr \frac{h_6^i}{r} (B_z - \frac{nkr}{m} B_\theta) h_j^1 + \int_{\Omega} dr \hat{h}_6^i (\frac{2nk}{mr} B_\theta h_j^1). \quad (3.62) \]

We now recall the remaining linearized MHD equations, starting from the second equation from top to bottom, the non-diagonal matrix subblocks \( A_{ji}(\alpha, \beta) \) read

\[ A_{ji}(2,4) = \int_{\Omega} dr \hat{h}_4^i \frac{m}{r} h_j^2, \quad (3.63) \]

\[ A_{ji}(2,5) = \int_{\Omega} dr \hat{h}_5^i \left( \frac{B_\theta}{r} + B_\theta \right) h_j^2 + \int_{\Omega} h_j^i \left( -\frac{nkr}{m} B_z \right) h_j^2, \quad (3.64) \]

\[ A_{ji}(2,6) = \int_{\Omega} dr \hat{h}_6^i \left[ \frac{n^2k^2r}{m} + \frac{m}{r} \right] h_j^2, \quad (3.65) \]

\[ A_{ji}(3,4) = \int_{\Omega} dr \hat{h}_4^i \frac{nk}{r} h_j^3, \quad (3.66) \]

\[ A_{ji}(3,5) = \int_{\Omega} dr \hat{h}_5^i \frac{nkB_\theta}{m} h_j^3 + \int_{\Omega} h_j^i \frac{B_\theta}{r} h_j^3, \quad (3.67) \]

\[ A_{ji}(3,6) = \int_{\Omega} dr \hat{h}_6^i \left[ -\frac{m}{r^2} - \frac{n^2k^2}{m} \right] B_\theta h_j^3, \quad (3.68) \]

\[ A_{ji}(4,1) = \int_{\Omega} dr \hat{h}_1^i \left[ -\frac{p_0}{r} \right] h_j^4 + \int_{\Omega} dr \hat{h}_1^i \left[ -\frac{\gamma p_0}{r} \right] h_j^4, \quad (3.69) \]
\[ A_{ji}(4, 2) = \int_{\Omega} dr h_i^2 \left[ -\frac{\gamma \rho_0 m}{r} \right] h_j^4, \quad (3.70) \]
\[ A_{ji}(4, 3) = \int_{\Omega} dr h_i^3 \left[ -\frac{\gamma \rho_0 n k}{t} \right] h_j^4, \quad (3.71) \]
\[ A_{ji}(5, 1) = \int_{\Omega} dr h_i^1 \left[ -\frac{m}{r} B_\theta - n k B_z \right] h_j^5, \quad (3.72) \]
\[ A_{ji}(5, 5) = h_i^5 \left[ \frac{\eta_0}{\eta} \right] \partial \Omega + \int_{\Omega} dr h_i^5 \left[ -\eta_0 \right] h_j^5 + \int_{\Omega} dr h_i^5 \frac{\eta_0}{\eta} h_j^5 \]
\[ + \int_{\Omega} dr \eta_0 h_i^5 \left[ -\frac{m^2}{r^2} - n^2 k^2 \right] h_j^5, \quad (3.73) \]
\[ A_{ji}(5, 6) = \int_{\Omega} dr h_i^6 \left[ -\frac{2 n k \eta_0}{r} \right] h_j^5, \quad (3.74) \]
\[ A_{ji}(6, 1) = \int_{\Omega} dr h_i^1 \left[ -B_z \right] h_j^6 + \int_{\Omega} dr h_i^1 \left[ -B'_z \right] h_j^6, \quad (3.75) \]
\[ A_{ji}(6, 2) = \int_{\Omega} dr h_i^2 \left[ -m B_z \right] h_j^6, \quad (3.76) \]
\[ A_{ji}(6, 3) = \int_{\Omega} dr h_i^3 \left[ \frac{m B_\theta}{r} \right] h_j^6, \quad (3.77) \]
\[ A_{ji}(6, 6) = \left[ \eta_0 \left( h_i^6 \frac{h_i^6}{r} \right) \right]_{\partial \Omega} + \int_{\Omega} dr h_i^6 \left[ -\eta_0 \right] h_j^6 \]
\[ + \int_{\Omega} dr h_i^6 \left[ \frac{\eta_0}{r} \right] h_j^6 + \int_{\Omega} dr h_i^6 \left[ -\frac{m^2}{r^2} - n^2 k^2 \right] h_j^6. \quad (3.78) \]

For the matrix \( B \) all the non-diagonal elements of a subblock are zero, whereas the diagonal elements are different from zero. Again, from the first equation of the linearized MHD to the bottom of them, the subblocks read

\[ B_{ji}(1, 1) = \int_{\Omega} dr h_i^1 \frac{\rho_0}{r} h_j^1, \quad (3.79) \]
\[ B_{ji}(2, 2) = \int_{\Omega} dr h_i^2 \frac{\rho_0}{r} h_j^2, \quad (3.80) \]
\[ B_{ji}(3, 3) = \int_{\Omega} dr h_i^3 \frac{\rho_0}{r} h_j^3, \quad (3.81) \]
\[ B_{ji}(4, 4) = \int_{\Omega} dr h_i^4 \frac{1}{r} h_j^4, \quad (3.82) \]
3.2 Numerical model

\[ B_{ji}(5,5) = \int_{\Omega} dr \, h^5_i \, h^5_j, \quad (3.83) \]

\[ B_{ji}(6,6) = \int_{\Omega} dr \, h^6_i \, h^6_j. \quad (3.84) \]

**Implementation of the resistive boundary conditions**

The boundary conditions for the resistive MHD are implemented as follows. First, we recall the surface terms appearing in the momentum equation which contains the subblocks \( A_{ji}(1,4), A_{ji}(1,5) \) and \( A_{ji}(1,6) \), i.e. Eqs. (3.60)-(3.61):

\[ W_{Mom} \equiv - \left[ h^1_j \frac{1}{r} h^4_i \right]_{\partial\Omega} - \left[ h^5_i \frac{B_\theta}{m} h^1_j \right]_{\partial\Omega} - \left[ \frac{h^6_i}{r} \left( B_z - \frac{nkr}{m} B_\theta \right) h^1_j \right]_{\partial\Omega}. \quad (3.85) \]

which is independent of the resistivity. The above expression can be refurnished, first we recall the divergence condition for the perturbed magnetic field, viz. \(-\frac{1}{r} (b'_1 - mb_2 - nb_3) = 0\), and we replace \( b'_1 \) in the above expression. After some algebra and expressing all in terms of \( \{ \hat{p}, v, b, B \} \), we get

\[ W_{Mom} = -v_1 \left( \frac{\hat{p}}{r} + b_2 B_\theta + \frac{b_3}{r} B_z \right). \quad (3.86) \]

From this expression, it is straighforward to express \( W_{Mom} \) as

\[ W_{Mom} = - (v \cdot n) \left( p + B \cdot b \right), \quad (3.87) \]

here \( n \) is the unit vector pointing in the radial direction. In particular this way of writing the b.c. makes life easier since it allows to compute directly the desired interface.

In the same fashion, the b.c. concerning a purely surface term emerges from the induction equation. This includes the subblocks \( A_{ji}(5,5) \) and \( A_{ji}(6,6) \) in Eqs. (3.73) and (3.78)

\[ W_{ind} = \left[ h^5_i \eta_0 h^5_j \right]_{\partial\Omega} + \left[ \eta_0 \left( h'^6_i - \frac{h^6_i}{r} \right) h^6_j \right]_{\partial\Omega}. \quad (3.88) \]

Again we transform to the original variable for the perturbed magnetic field \( b \) and the equation is rewritten as

\[ W_{ind} = \eta_0 \frac{1}{2} \left( b_z^2 - b_r^2 \right) - \eta_0 r b_r^2. \quad (3.89) \]
3. Physical and numerical models

3.2.4 A crucial decision: selection of basis $h(r)$

Having obtained the matrix subblocks $A$ and $B$ for our numerical problem, we now define which basis are suitable for expanding the 6 state variables. Stability analysis for the ideal linear MHD equations in cylindrical geometry shows that the full spectrum is not well reproduced if only linear finite elements are used [45]. Thus, a first warning appears regarding the sensitivity of this numerical model to the basis expansion functions. The failure in this attempt is that an eigenvalue, associated to a particular eigenfunction, instead of being infinitely degenerated, it shows a discrete spectrum which extended to infinity. This is known as spectrum pollution.

Nevertheless, the pollution is avoided by choosing different dependencies on the basis functions for the states variables [46, 47]. In resistive MHD, it is addressed in [37] that linear elements for $\nu_1, b_1, b_3$ and piecewise constant elements for $\nu_2, \nu_3, \hat{p}$ yields to a poor numerical approximation of the spectrum. To overcome this, the introduction of higher order elements are required and used. One possible way to proceed is to use cubic Hermite elements for $\nu_1$ and $b_1$, whereas quadratic finite elements are implemented for $\nu_2, \nu_3, \hat{p}$, and $b_3$ [48, 37]. In either case, two orthogonal functions define a complete set of expansion. Thus, for the radial components of both velocity and magnetic field, the basis are

$$H_{i,1}^k(r) = \begin{cases} \left( \frac{r - r_{i-1}}{r_i - r_{i-1}} \right)^2 \left( 3 - 2 \frac{r - r_{i-1}}{r_i - r_{i-1}} \right), & \text{if } r \in [r_{i-1}, r_i], \\ \left( \frac{r_{i+1} - r}{r_{i+1} - r_i} \right)^2 \left( 3 - 2 \frac{r_{i+1} - r}{r_{i+1} - r_i} \right), & \text{if } r \in [r_i, r_{i+1}], \\ 0 & \text{elsewhere,} \end{cases} \quad (3.90)$$

and

$$H_{i,\Pi}^k(r) = \begin{cases} \frac{r - r_i}{r_{i-1} - r_i} \left( \frac{r - r_{i-1}}{r_i - r_{i-1}} \right)^2, & \text{if } r \in [r_{i-1}, r_i], \\ \frac{r - r_{i+1}}{r_{i+1} - r_i} \left( \frac{r - r_i}{r_{i+1} - r_i} \right)^2, & \text{if } r \in [r_i, r_{i+1}], \\ 0 & \text{elsewhere,} \end{cases} \quad (3.91)$$

which will be valid only for $k = 1, 5$. Again, we have used the superscripts’ notation of the state vector $u(r)$, viz.

$$(\nu_1, \nu_2, \nu_3, \hat{p}, b_1, b_3) \equiv (\nu^1, \nu^2, \nu^3, \nu^4, \nu^5, \nu^6).$$

Therefore, the expansion used to approximate $u^k(r)$ with Hermite spline functions is written as

$$u^k(r) \approx \hat{u}^k(r) = \sum_{i=0}^{N} a_{i,1}^k H_{i,1}^k(r) + a_{i,\Pi}^k H_{i,\Pi}^k(r) \quad \text{for } k = 1, 5. \quad (3.92)$$

Note that the above expansion is nothing more than Eq.(3.57) of last section.
3.2 Numerical model

On the other hand, for the other state variables, the quadratic basis functions are

\[
Q^k_{i,1}(r) = \begin{cases} 
\frac{(2r - ri - ri-1)}{(ri - ri-1)^2} (r - ri), & \text{if } r \in [ri-1, ri], \\
\frac{(2r - ri+1 - ri)}{(ri+1 - ri)^2} (r - ri+1), & \text{if } r \in [ri, ri+1], \\
0 & \text{elsewhere},
\end{cases}
\]  

(3.93)

and

\[
Q^k_{i,II}(r) = \begin{cases} 
\frac{4(r - ri-1)(ri - r)}{(ri - ri-1)^2}, & \text{if } r \in [ri-1, ri], \\
0 & \text{elsewhere},
\end{cases}
\]  

(3.94)

for \( k = 2, 3, 4, 6 \). For completeness, the expansion \( \tilde{u}^k(r) \) of the remaining state variables reads

\[
\tilde{u}^k(r) \approx \tilde{u}^k(r) = \sum_{i=0}^{N} a^k_{i,1} Q^k_{i,1}(r) + a^k_{i,II} Q^k_{i,II}(r) \quad \text{for } k = 2, 3, 4, 6.
\]

(3.95)

Both cubic and quadratic elements are schematically shown in figure 3.7. To summarize the actual numerical method, first we need to discretize the domain (either uniformly or not), we expand the state vector \( \mathbf{u}^T \) in terms of cubic and quadratic basis elements followed by a minimization in some average of the residual function \( R_{\tilde{u}}(r) \). Finally, we invoke the weak formulation of the Galerkin method which gives rise to a general EVP problem. In the following chapter it is presented the application of the numerical code of the resistive cylindrical plasma column.

\[ ^7 \text{However, in [45] the displacement vector } \xi \text{ approach is used.} \]
3. Physical and numerical models
Chapter 4

Results & discussion

In this section the application of the code for both ideal and resistive MHD is presented. Results of the full ideal spectrum, perturbed profiles and growth rates together with the change of resistivity are shown and discussed. All simulations were performed in Matlab environment.

4.1 Ideal case

First, we perform simulations that reproduce known ideal MHD results which in turn will give validation to our current numerical model. To do so, we assume an homogeneous plasma surrounded by a perfect conducting wall. In addition, it is considered a constant longitudinal magnetic field component $B_z$ and a linearly increasing poloidal magnetic field component $B_\theta$,

$$B_z(r) = 1, \quad B_\theta = cr, \quad c = \text{const.},$$

$$\rho = 1,$$  \hspace{1cm} (4.1)

so that the current density is evenly distributed all over the plasma and a according to the equilibrium equation, whereas the pressure distribution has a parabolic profile

$$j_z = \frac{\nabla \times B_0}{2c},$$

$$p = \frac{1}{(r^2 B_0)} \frac{1}{c^2} \frac{1}{(1 - r^2)},$$ \hspace{1cm} (4.2)

it turns out that with these profiles, the safety factor $q(r)$ (see Eq. (3.25) in Section 3.1.3) reads

$$q(r) = \frac{rkB_z}{B_\theta} = \frac{k}{c},$$ \hspace{1cm} (4.3)

i.e. $q(r)$ is constant along the plasma and depends on the ratio between the already defined periodicity length $k = 2\pi/L$ and the $c$ constant (its value typically is less than unity). Moreover, the plasma with the above equilibrium
profiles and bcs: (1) the external modes are mitigated and (2) the plasma is unstable against the $m=1$ interchange mode if $q=1$ for values of $r$ such that $r < a$ [37, 49]. This unstable situation can be avoided if the right combination of $k/c$ is given, thus assuring that $q > 1$ over the whole plasma radius. In the performed simulation the following values were used: $c = 0.1$ and $k = 0.2$. Once the $m=1$ unstable mode is overcome, the $m=2$ interchange mode is then the most dangerous instability. Figure 4.1 shows the full structure of the ideal MHD spectrum in presence of the $m=2$ instability.

![Figure 4.1](image)

**Figure 4.1:** Complete spectrum of the ideal MHD problem in presence of the $m=-2$ instability. Top panel: the stable part of the spectrum characterized by 3 branches: fast, Alfvén and slow modes. Bottom panel: the eigenvalue $i\omega$ results purely imaginary and thus an unstable wave is established.

The ideal MHD spectrum presents three frequency branches, namely the fast magnetoacoustic waves, the Alfvén and the slow modes. In the presence of the $m=2$ instability the slow mode degenerates to one point $\omega_S = 0$ [37, 50]. Therefore, it deserves particular attention the limit point $\omega \to 0$, since Suydam’s criterion is violated (Eq. (3.26) in Section 3.1.3) theory assures that the marginal point $\omega = 0$ is an accumulation point (or cluster) of the unstable side of the discrete spectrum [51, 37] (bottom panel of figure 4.1). This is a consequence of a vanishing parallel gradient operator $F \equiv mB_\theta/r + kB_z$ that
4.1 Ideal case

appears in the equation of motion of the ideal MHD equations\(^1\) (see Section 3.2.3). The operator \(F\) plays an important role in stability analysis since it represents the projection of the gradient operator parallel to the magnetic field, i.e. \(F = -i \mathbf{B} \cdot \nabla = k_0 \cdot \mathbf{B}^2\). Hence, if \(F\) vanishes at a particular location inside the plasma, means that the stabilizing magnetic field line bending energy vanishes there as well, favouring an interchange instability.

The accuracy of the full spectrum strongly depends on the number of radial grid points used in the basis expansion defined on Section 3.2.3 together with the number of points used in the numerical integration\(^3\). For instance, the number of radial grid points \(N\) needed to describe accurately the accumulation point at \(\omega = 0\) should be at least \(N > 60\). Several test cases varying the number of radial and integration points were performed. The critical value for an optimal simulation is defined by the pair \((60, 30)\), where the first element inside the braquets defines the number of radial grid points, whereas the second one the number of integration points in the quadrature formula. Poor simulation is achieved for less than 20 grid points, since the frequency point succession towards the cluster point, on the unstable part of the spectrum, is no longer well reproduced. A couple of these test cases are shown in figure 4.2.

\[\text{Figure 4.2: Complete spectrum of the ideal MHD problem in presence of the } m = -2 \text{ instability. In what follows } (N_{\text{rad}}, G_{\text{quad}}) \text{ represents the pair: radial grid and quadrature integration points, respectively. (a) Case (90, 10): Acceptable marginal point, no symmetry on unstable part. (b) Case (20, 20) The marginal point is not well established since there is no sequence towards the cluster point.}\]

\(^1\)Probably this model, the incompressible homogeneous plasma cylinder with constant safety factor, was first considered by H. Alfvén to study unstable loops as a mechanism for the generation of cosmic magnetic fields [52].

\(^2\)Here, the parallel wave vector, \(k_0\), is given by \(k_0 \equiv (m/r,k)\). By parallel it is meant that \(k_0\) points in the same direction of the magnetic field \(B_0\).

\(^3\)Also known as quadrature formulas which need to reach a particular order of accuracy.
4. Results & discussion

As explained in Section 3.2, in the radial direction two kinds of finite elements per grid point are used to avoid spectral pollution (see Eq. (3.57) in Section 3.2.3 and Figure 3.7 in Section 3.2.4). The latter together with the 6 components of the state vector \( u^T \), makes the total number of unknowns \( 2 \times 6 \times N \), with \( N \) the number of radial grid points as mentioned before. For instance, the 2 basis expansion functions makes the sub-blocks apperaing in our EVP, viz. \( A_a = i \omega B_a \), matrices of dimension \( 12 \times 12 \). Therefore, an algorithm is required to solve this matricial problem. Among the available algebraic methods, the Cholesky factorization was used for the ideal MHD simulations, due to the Hermiticity properties of the matrices. This decomposition method is chosen because it is about a factor of two faster than alternative methods (such as \( LU \) factorization) [53].

4.2 Resistive case

In this section the application of our FEM code to the homogeneous and resistive cylindrical plasma column surrounded by a perfect conducting wall is presented. With the introduction of finite (constant) resistivity, the magnetic topology is not conserved, i.e. the magnetic field is no longer frozen into the fluid. This gives rise to internal resistive layers which introduce new modes of instability.

These modes can be: (1) pressure-driven such as the interchange instability, (2) current-driven such as the tearing modes or (3) both. Regarding the tearing mode, it decouples the fluid and the magnetic field forming an island configuration around resonant surfaces [37, 54]. Their growth rates, \( \lambda_r \equiv \text{Re}(i \omega) \), and its functional dependence on the resistivity \( \eta \) is studied. To do so, we establish a tokamak-like equilibrium profiles

\[
\begin{align*}
  j_z &= j_0 \left( 1 - r^2/a^2 \right)^\nu, \\
  B_z &= 1, \\
  \rho &= 1.
\end{align*}
\]

Once the profiles are given, the safety factor is calculated

\[
q(r) = q_0 \frac{(\nu + 1) r^2/a^2}{1 - (1 - r^2/a^2)^{\nu+1}},
\]

where \( q_0 \) is given by \( j_0 = 2kq_0 \), where \( k = 2\pi/L \) is the periodicity length of the cylinder as before. This provides the possibility to adjust \( q_0 \) by means of \( j_0 \) on the one hand, whereas the overall shape of the \( q \)-profile can be modified in terms of the parameter \( \nu \)

\[
\frac{q(a)}{q(0)} = \nu + 1,
\]
4.2 Resistive case

on the other hand. Again, Suydam’s criterion is violated, thus the equilibrium defined in Eqs. (4.4)-(4.6) is known to be unstable against the $m = 1$ tearing mode if $q = 1$ is located inside the plasma. Should this condition is overcome, the following most unstable mode is the $m = 2$ tearing mode for $2.20 \leq q(a) \leq 4.00$ [37, 55, 49]. In the performed simulations, we set $\nu = 2$ and the perturbation’s mode and wave numbers are: $n = 1$, $m = -2$, and $k = 0.1$. The growth rate of associated to this mode is plotted as function of the safety factor at the edge $q(a)$ in figure 4.3.

![Figure 4.3: Growth rates of tearing mode as a function of the total current. The growth rate becomes much smaller as $q(a)$ approaches to the value 2.20 in agreement with the reported results in [37, 55, 49].](image)

The value $j_0$ is chosen such that the resonant surface $r_s$ lies in the middle of the plasma which means that the wave vector of the perturbation is perpendicular to the magnetic field in that location, i.e. $F = k \cdot B_0|_{r=r_s} = 0$ with $q(r_s) = 2$ and $r_s = 0.5^4$. For numerical resolution near the resonant surface, it was developed a grid pack that accumulates mesh points (radial grid points) around $r_s$ according to a normalized distribution function. In the present simulation it has been used a Gaussian-like distribution function around $r_s = 0.5$. Throughout this work it was assumed a vanishing perturbed

---

4Recall the fact that dimensionless parameters are taken into account. In terms of the plasma minor radius: $r_s = 0.5a$. 
resistivity $\eta_1 \equiv 0$, thus the rippling mode is eliminated since the convective law [56] $\eta_1 = -(v/\gamma) \cdot \nabla \eta_0 = 0$. Also, the gravitational interchange modes are not considered since the mass density gradient is absent in the description of the system. For more realistic situations these terms should be included in the resistive code, adding the corresponding terms in the Ohm’s law and in the equation of motion.

In spite of the mentioned simplifications, the resistive code finds both growth rates associated to the current-driven tearing mode and the pressure-driven interchange mode. Analytic theory gives scaling laws of the growth rates with resistivity for both the pressure and current instabilities. For small resistivity values, the former exhibits the scaling $\lambda_{\text{tear}} \propto \eta^{3/5}$, whereas the latter scales as $\lambda_{\text{inter}} \propto \eta^{1/3}$. It was studied the growth rates in the following range of resistivity values $\eta \in [10^{-10}, 10^{-5}]$. Those were the upper and lower limit $\eta$ values for our studies, further values were not considered. Figure 4.4 shows the dependence of these modes with resistivity.

The most unstable mode, tearing mode, scales with $\eta^{3/5}$ for small values of the resistivity and saturates for values greater than $\eta = 10^{-5}$. The second unstable mode, viz. resistive interchange, scales like $\eta^{1/3}$. If an accurate simulation is desired, then it is crucial the selected number of radial points $N$. A
minimum of 200 mesh points is the threshold for an accurate result, otherwise non physical behaviour is found at the lower values of $\eta$ for both growth rates. The latter was tested for the set of radial points $N = \{30, 50, 70, 100, 150\}$. Additional test cases were performed to study growth rates dropping the value $\eta = 10^{-10}$ to smaller ones, say $10^{-11}$ or $10^{-12}$. It was found a knee on the growth rates (for both tearing and interchange modes) which changes abruptly the slopes of the calculated results\(^{5}\). On this basis, it was established that reliable results will be considered on the interval $\eta \in [10^{-10}, 10^{-5}]$\(^{6}\).

Once the selection of the number of mesh points yields an acceptable prediction of the growth rates, we now focus our attention on the global structure of the perturbed profiles. In what follows a series of figures 4.5-4.7 display the modification of the profiles with the values of the resistivity.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure4.5.png}
\caption{Radial velocity perturbation $v_1(r)$ versus the normalized radius $r/a$.}
\end{figure}

Figures 4.5-4.6 show the highly localized structure of the velocity profiles around the resonant surface $r_s = 0.5$ for different values of $\eta$ (values differ in orders of magnitude). It is worth noting that for all values of $\eta$ the velocity profiles have a finite value at the resonant surface and it was evident the presence of a radial node in the velocity profiles for resistivity values $\eta < 10^{-6}$.

The application of the grid pack is depicted in figure 4.7 where the mesh of $N = 200$ radial points and gaussian like distribution function of radial grid points is deployed. Enlarge view is done in figure 4.7(b) to show the mesh point accumulation around the resonant surface $r_s = 0.5$. At this particular location,

\(^5\)This applies to all cases with different radial points.

\(^6\)Saturation is found for the upper bound, which does not represent any problem.
an increase in numerical resolution was found as more radial points are taken into account together with an appropriate distribution function’s width. A set of simulations were performed combining the following pair of radial points and distribution’s width, viz. \( \{N_{rad}, w_{width}\} \): \( N = \{30, 50, 70, 100, 150\} \) and \( w_{width} = \{0.1, 0.01, 0.001\} \).

Regarding the structure of the magnetic field profiles with the resistivity, figure 4.7, it was found a less singular behaviour around the resonant point for the radial component \( b_1 \) for all values of resistivity. However the poloidal and toroidal components, \( b_2 \) and \( b_3 \), are highly localized and exhibit very singular structure for small values of resistivity \( \eta \in [10^{-10}, 10^{-7}] \), they thus present large gradients around the \( r_s = 0.5 \).
4.2 Resistive case

(a) Perturbed radial velocity $v_1 = r v_r$ for $\eta = 10^{-8}$ in arbitrary units.

(b) Perturbed radial velocity $v_1 = r v_r$ for $\eta = 10^{-5}$ in arbitrary units.

**Figure 4.6**: Eigenfunctions corresponding to the perturbed radial velocity $v_1$ for two different values of resistivity: (a) $\eta = 10^{-8}$ and (b) $\eta = 10^{-5}$ for the case of the resistive cylindrical plasma in direct contact with a perfect conducting wall.
4. Results & discussion

(a) Perturbed magnetic field $b_1$ for $\eta = 10^{-10}$ in arbitrary units. Mesh points are accumulated around the resonant surface via a grid pack.

(b) Enlarged scale: perturbed magnetic components profiles $b_1 = (b_1, b_2, b_3)$ for $\eta = 10^{-10}$.

**Figure 4.7**: Eigenfunctions corresponding to the 3 components of the perturbed magnetic field $b_1$ for $\eta = 10^{-10}$: (a) Complete domain range $r/a \in [0, 1]$ and (b) Enlarged view of domain $r/a \in [0.45, 0.55]$ for the resistive cylindrical plasma in direct contact with a perfect conducting wall.
Solving the resistive MHD EVP differs from the ideal calculation, because the matrices appearing in the EVP are not Hermitian anymore (due to the introduction of finite resistivity $\eta \neq 0$), giving rise to a complex eigenvalue problem. To my concern, the problems involving non-hermitian operators have no unique approach. Typically one may choose either direct or iterative methods to solve numerically large systems; the QR method, belonging to the direct ones, is used in our resistive MHD EVP.

The QR algorithm is widely used due to its reliability and stable properties [57, 37]. The method is based on a decomposition of the original matrix with triangular and orthogonal matrices and the algorithm consists of a succession of orthogonal transformations [58]. In the QR algorithm the total number of operations is proportional to $n^3$ for a $n \times n$ matrix. The main drawback of the QR algorithm is that it gives rise to full matrices and thus restricting the spatial resolution due to the memory requirements, i.e. can be computationally expensive for large systems. However, improvements to these method has been developed, thus making it even more effective [59]. But, should spatial refinement and non-expensive factorization are required, alternative algorithms such as the inverse iteration and Jacobi-Davidson methods are available. In the case of the former, it does not present any advantage regarding the number of operations needed, since it is also proportional to $n^3$.

Although the method is quite accurate to calculate the growth rates of the most unstable internal modes for the fixed boundary plasma case (plasma-perfect conducting wall), simulations involving the $m = 1$ external kink mode are also performed. A careful study of the outcome showed that the calculation of the eigenfunctions turns out to be very sensible regarding the regularity condition on axis. It is worth saying that the growth rate resembles previous results [37, 60, 61], but improvement of numerics is still needed. The accuracy of this test case (plasma-vacuum) could be related to the selection of the algorithm in the calculation of the EVP.
4. Results & discussion
Chapter 5

Summary & conclusions

On the development of a full eigenproblem solver, several test cases should be performed before implementation on fusion machines’ feedback control systems. In this work it is presented a resistive MHD code that allows us to study the effect of resistivity on the cylindrical plasma model.

The numerical approach concerns non-trivial discretization of general matrix operators and the solution of complex eigenvalue problem via the Galerkin method and finite elements. Careful selection of basis expansion functions in the state vector avoids spectral pollution in calculations. Two case studies were performed, namely the ideal and resistive MHD models within the plasma-perfect conducting wall interface.

For ideal MHD case, the full spectrum is accurately determined where the three frequency branches are distinguished: the magnetoacoustic waves, Alfvén and slow modes. Our results are in agreement with the marginal stability $\omega \rightarrow 0$ of the spectrum where the parallel gradient operator $\mathbf{F} = \mathbf{k} \cdot \mathbf{B}$ vanishes. It turns out that the optimal selection of grid points ($N > 60$) together with an appropriate matrix calculation’s algorithms (Cholesky) gives rise to accurate results.

The resistive MHD simulations revealed several features on the calculation of the growth rates and perturbed eigenfunctions. First, the simulations were based on tokamak-like equilibrium profiles for $\rho$, $\mathbf{B}(r)$, $\mathbf{j}(r)$ and $q(r)$ in the presence of the $n = 1$, $m = -2$ instability. The particular selection of these profiles allow us to deploy and test the grid pack around the resonant surface (located in the middle of the plasma radius) increasing the local resolution in that region. Second, the code is capable to obtain the growth rates for the two unstable internal modes, namely the tearing current-driven and interchange pressure-driven modes. In accordance to theoretical results, it was found growth rates’ scaling relations with resistivity, viz. $\lambda_{\text{tear}} \propto \eta^{3/5}$ and $\lambda_{\text{inter}} \propto \eta^{1/3}$. However, we can only assure this for small values of $\eta$. In addition, it was found high localized structure of the eigenfunctions near the resonant surface. This structure becomes less singular as the value of resistivity increases.

Improvements should be done in forthcoming simulations. For instance,
the associated growth rate to the external mode $m = -1$ is not well reproduced. It is not clear whether the regularity condition on axis or the method of matrix calculation (or both) affect the final outcome. The accurate modelling of this external mode is particularly important if extensions of the code into more realistic situations are desired, e.g. the introduction of resistive wall boundary condition and its effects on the plasma. Although the cylindrical plasma model lacks toroidal effects, e.g. Shafranov shift, the code capture the essential physics of the resistivity effects for further application on EXTRAP T2R. Potential improvements such as resistivity that depends on position and the introduction of the viscosity in the equation of motion are highly desirable.
Appendix A

Vector and coordinates

A.1 Vector identities

Let \( \mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3 \) and \( \Phi \) an scalar function of the space variables \( \mathbf{r} \). The following list shows the most frequently exploited identities in this work:

\[
\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}),
\]

(A.1)

\[
\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b} (\mathbf{a} \cdot \mathbf{c}) - \mathbf{c} (\mathbf{a} \cdot \mathbf{b})
\]

(A.2)

\[
\nabla \times \nabla \Phi = 0,
\]

(A.3)

\[
\nabla \cdot (\nabla \times \mathbf{a}) = 0,
\]

(A.4)

\[
\nabla \times (\nabla \times \mathbf{a}) = \nabla (\nabla \cdot \mathbf{a}) - \nabla^2 \mathbf{a},
\]

(A.5)

\[
\nabla \cdot (\Phi \mathbf{a}) = \mathbf{a} \cdot \nabla \Phi + \Phi \nabla \cdot \mathbf{a},
\]

(A.6)

\[
\nabla \times (\Phi \mathbf{a}) = \nabla \Phi \times \mathbf{a} + \Phi \nabla \times \mathbf{a},
\]

(A.7)

\[
\mathbf{a} \times (\nabla \times \mathbf{b}) = (\nabla \mathbf{b}) \cdot \mathbf{a} - \mathbf{a} \cdot \nabla \mathbf{b},
\]

(A.8)

\[
(a \times \nabla) \times \mathbf{b} = (\nabla \mathbf{b}) \cdot \mathbf{a} - \mathbf{a} (\nabla \cdot \mathbf{b}),
\]

(A.9)

\[
\nabla (\mathbf{a} \cdot \mathbf{b}) = (\nabla \mathbf{a}) \cdot \mathbf{b} + (\nabla \mathbf{b}) \cdot \mathbf{a}
\]

\[= \mathbf{a} \cdot \nabla \mathbf{b} + \mathbf{b} \cdot \nabla \mathbf{a} + \mathbf{a} \times (\nabla \times \mathbf{b}) + \mathbf{b} \times (\nabla \times \mathbf{a}),
\]

(A.10)

\[
\nabla (ab) = a \cdot \nabla b + b \nabla \cdot a,
\]

(A.11)
A. Vector and coordinates

\[ \nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot \nabla \times \mathbf{a} - \mathbf{a} \cdot \nabla \times \mathbf{b}, \tag{A.12} \]

\[ \nabla \times (\mathbf{a} \times \mathbf{b}) = \nabla \cdot (\mathbf{b} \mathbf{a} - \mathbf{a} \mathbf{b}) \]

\[ = \mathbf{a} \nabla \cdot \mathbf{b} + \mathbf{b} \nabla \cdot \mathbf{a} - \mathbf{b} \nabla \cdot \mathbf{a} - \mathbf{a} \nabla \cdot \mathbf{b}, \tag{A.13} \]

\[ \iiint \nabla \cdot \mathbf{a} \, d\tau = \iint \mathbf{a} \cdot \mathbf{n} \, d\sigma \quad (Gauss), \tag{A.14} \]

\[ \mathbf{a} \rightarrow \mathbf{a} \times \mathbf{c} \text{(const)} \implies \iiint \nabla \times \mathbf{a} \, d\tau = \iint \mathbf{n} \times \mathbf{a} \, d\sigma, \tag{A.15} \]

\[ \mathbf{a} \rightarrow \Phi \mathbf{c} \text{(const)} \implies \iiint \nabla \Phi \, d\tau = \iint \Phi \mathbf{n} \, d\sigma, \tag{A.16} \]

\[ \mathbf{a} \rightarrow \Phi \nabla \Psi - \Psi \nabla \Phi \implies \]

\[ \iiint (\Phi \nabla^2 \Psi - \Psi \nabla^2 \Phi) \, d\tau = \iint (\Phi \nabla \Psi - \Psi \nabla \Phi) \cdot \mathbf{n} \, d\sigma \quad (Green), \tag{A.17} \]

\[ \iint (\nabla \times \mathbf{a}) \cdot \mathbf{n} \, d\sigma = \oint \mathbf{a} \cdot d\mathbf{l} \quad (Stokes), \tag{A.18} \]

\[ \mathbf{a} \rightarrow \mathbf{a} \times \mathbf{c} \text{(const)} \implies \iint (\mathbf{n} \times \nabla) \times \mathbf{a} \, d\sigma = \oint d\mathbf{l} \times \mathbf{a}, \tag{A.19} \]

\[ \mathbf{a} \rightarrow \Phi \mathbf{c} \text{(const)} \implies \iint \mathbf{n} \times \nabla \Phi \, d\sigma = \oint \Phi \, d\mathbf{l}. \tag{A.20} \]

A.2 Vector expressions in orthogonal coordinates

Let the position vector \( \mathbf{r} \) of a particle be a function of orthogonal coordinates \( x_i \),

\[ \mathbf{r} = \mathbf{r}(x_1, x_2, x_3) \iff \begin{cases} x = x(x_1, x_2, x_3) \\ y = y(x_1, x_2, x_3) \\ z = z(x_1, x_2, x_3) \end{cases} \tag{A.21} \]
the following geometric quantities are associated to such transformation, namely

\[ h_i \equiv \left| \frac{\partial \mathbf{r}}{\partial x_i} \right| \quad (\text{scale factors}), \quad (A.22) \]

\[ \mathbf{e}_i \equiv \left( \frac{1}{h_i} \right) \frac{\partial \mathbf{r}}{\partial x_i}, \quad \mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} \quad (\text{dimensionless unit vectors}), \quad (A.23) \]

\[ d\ell = \sqrt{\sum_i (h_i \, dx_i)^2} \quad (\text{line element}), \quad (A.24) \]

\[ d\tau = h_1 h_2 h_3 \, dx_1 dx_2 dx_3 \quad (\text{volume element}). \quad (A.25) \]

In addition, the vector representation of \( \mathbf{A} \) is given by:

\[ \mathbf{A} = \sum_i \hat{A}_i \mathbf{e}_i \quad (A.26) \]

where the coefficients \( \hat{A}_i \) are the physical components with the same dimension as \( \mathbf{A} \).

Let \( \Psi \) be a scalar function of \( \mathbf{r} \) and \( \mathbf{A} \) a vector function of \( \mathbf{r} \). The differential operators are:

\[ \nabla \psi = \sum_i \frac{1}{h_i} \frac{\partial \psi}{\partial x_i} \mathbf{e}_i, \quad (A.27) \]

\[ \nabla^2 \psi = \frac{1}{h_1 h_2 h_3} \left[ \frac{\partial}{\partial x_1} \left( h_2 h_3 \frac{\partial \psi}{\partial x_1} \right) + \frac{\partial}{\partial x_2} \left( h_1 h_3 \frac{\partial \psi}{\partial x_2} \right) + \frac{\partial}{\partial x_3} \left( h_1 h_2 \frac{\partial \psi}{\partial x_3} \right) \right], \quad (A.28) \]

\[ \nabla \cdot \mathbf{A} = \frac{1}{h_1 h_2 h_3} \left[ \frac{\partial}{\partial x_1} \left( h_2 h_3 \hat{A}_1 \right) + \frac{\partial}{\partial x_2} \left( h_1 h_3 \hat{A}_2 \right) + \frac{\partial}{\partial x_3} \left( h_1 h_2 \hat{A}_3 \right) \right], \quad (A.29) \]

\[ \nabla \times \mathbf{A} = \frac{1}{h_2 h_3} \left[ \frac{\partial}{\partial x_2} \left( h_3 \hat{A}_3 \right) - \frac{\partial}{\partial x_3} \left( h_2 \hat{A}_2 \right) \right] \mathbf{e}_1 \\
+ \frac{1}{h_1 h_3} \left[ \frac{\partial}{\partial x_3} \left( h_1 \hat{A}_1 \right) - \frac{\partial}{\partial x_1} \left( h_3 \hat{A}_3 \right) \right] \mathbf{e}_2 \\
+ \frac{1}{h_1 h_2} \left[ \frac{\partial}{\partial x_1} \left( h_2 \hat{A}_2 \right) - \frac{\partial}{\partial x_2} \left( h_1 \hat{A}_1 \right) \right] \mathbf{e}_3 \quad (A.30) \]
A.2.1 Cylindrical coordinates \((r, \theta, z)\)

From figure A.1, the components of the vector position \(\mathbf{R}\) and the scale factors are given by

\[
\begin{cases}
  x = r \cos \theta \\
y = r \sin \theta \\
z = z
\end{cases}
\]

Leading to

\[ h_1 = 1, \quad h_2 = r, \quad h_3 = 1. \quad (A.31)\]

As before, let \(\Psi\) be a scalar function of \((r, \theta, z)\) and \(\mathbf{A}\) a vector function of \((r, \theta, z)\). Using the relations \((A.27)-(A.30)\), the differential operators in this coordinate system are:

\[
\nabla \Psi = \frac{\partial \Psi}{\partial r} \mathbf{e}_r + \frac{1}{r} \frac{\partial \Psi}{\partial \theta} \mathbf{e}_\theta + \frac{\partial \Psi}{\partial z} \mathbf{e}_z, \quad (A.32) \\

\nabla^2 \Psi = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \Psi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \Psi}{\partial \theta^2} + \frac{\partial^2 \Psi}{\partial z^2}, \quad (A.33) \\

\nabla \cdot \mathbf{A} = \frac{1}{r} \frac{\partial}{\partial r} \left( r A_r \right) + \frac{1}{r} \frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z}, \quad (A.34) \\

\nabla \times \mathbf{A} = \left( \frac{1}{r} \frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z} \right) \mathbf{e}_r \\

+ \left( \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} \right) \mathbf{e}_\theta + \left( \frac{1}{r} \frac{\partial (r A_\theta)}{\partial r} - \frac{1}{r} \frac{\partial A_r}{\partial \theta} \right) \mathbf{e}_z, \quad (A.35) \\

\nabla^2 \mathbf{A} = \left( \nabla^2 A_r - \frac{1}{r^2} A_r - \frac{2}{r^2} \frac{\partial A_\theta}{\partial \theta} \right) \mathbf{e}_r \\

+ \left( \nabla^2 A_\theta - \frac{1}{r^2} A_\theta + \frac{2}{r^2} \frac{\partial A_r}{\partial \theta} \right) \mathbf{e}_\theta + \nabla^2 A_z \mathbf{e}_z, \quad (A.36) 
\]
\[ \nabla \times \nabla \times \mathbf{A} = \left[ -\frac{1}{r^2} \frac{\partial^2 A_r}{\partial \theta^2} - \frac{\partial^2 A_r}{\partial z^2} + \frac{1}{r^2} \frac{\partial^2 (r A_\theta)}{\partial \theta \partial r} + \frac{\partial^2 A_z}{\partial z \partial r} \right] \mathbf{e}_r \]

\[ + \left[ \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial A_r}{\partial \theta} \right) - \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial (r A_\theta)}{\partial r} \right) - \frac{\partial^2 A_\theta}{\partial z^2} + \frac{1}{r} \frac{\partial^2 A_z}{\partial z \partial \theta} \right] \mathbf{e}_\theta \]

\[ + \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{r \partial A_r}{\partial z} \right) + \frac{1}{r} \frac{\partial^2 A_\theta}{\partial \theta \partial z} - \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{r A_z}{\partial r} \right) - \frac{1}{r^2} \frac{\partial^2 A_z}{\partial \theta^2} \right] \mathbf{e}_z. \]

(A.37)

**Figure A.1**: Cylindrical coordinate system.
A. Vector and coordinates
Appendix B

A survey into kinetic theory

B.1 From the kinetic to the MHD model

Consider a plasma consisting of electrons and ions. Solving the coupled equation of motion for all particles including their interaction sounds a rather impossible task to achieve. Clearly a statistical description of the plasma is needed where the information of the individuality of the particles is lost but the relevant physical collective dynamics is retained. In the framework of such approach, all alike particles of species $\alpha$ is looked as an ensemble in six dimensional phase space $d^3r \, d^3v$ that is described by its distribution function $f_\alpha(r,v,t)$ for the electrons and the ions ($\alpha = e, i$) where

$$f_\alpha(r,v,t) \, d^3r \, d^3v = \text{Number of particles in } d^3r \, d^3v \text{ at time } t \quad (B.1)$$

The temporal evolution of the distribution function in phase space is described by the total change of $f_\alpha$ in space, velocity, and time variables

$$\frac{df_\alpha}{dt} = \frac{\partial f_\alpha}{\partial t} + \frac{\partial f_\alpha}{\partial r} \cdot \frac{dr}{dt} + \frac{\partial f_\alpha}{\partial v} \cdot \frac{dv}{dt}$$

$$= \frac{\partial f_\alpha}{\partial t} + v \cdot \nabla f_\alpha + \frac{q_\alpha}{m_\alpha} (E + v \times B) \cdot \nabla_v f_\alpha \quad (B.2)$$

where the expression (2.8) for the acceleration $dv/dt$ of the particles has been inserted in the last line. The following notation for spatial and velocity derivatives is used: $\partial/\partial r \equiv \nabla = (\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3)$ and analogous for the term $\partial/\partial v$ but in velocity space.

A more realistic situation is considered when interactions -such as collisions- between particles take place. If this is the case, the total change in time of the
distribution function of ions and electrons is given by the *Boltzmann equation*:

\[
\frac{\partial f_\alpha}{\partial t} + v \cdot \nabla f_\alpha + \frac{q_\alpha}{m_\alpha} (E + v \times B) \cdot \nabla_v f_\alpha = C_\alpha = \left( \frac{\partial f_\alpha}{\partial t} \right)_{\text{coll}} \tag{B.3}
\]

here the expression on the RHS of equation (B.3) is referred as the collision operator. It represents the rate of change of the distribution function due to short-range inter-particle interactions [38].

On the other hand, neglecting the collisional operator leads to the well-known *Vlasov equation* [62, 63]:

\[
\frac{\partial f_\alpha}{\partial t} + v \cdot \nabla f_\alpha + \frac{q_\alpha}{m_\alpha} (E + v \times B) \cdot \nabla_v f_\alpha = 0 \tag{B.4}
\]

which means that distribution function \( f_\alpha(r, v, t) \) will remain constant along the trajectory in phase space and thus it will be a conserved quantity.

To this end, a closed system of equations is needed. Therefore, the latter is found when it is coupled either the Boltzmann equation (B.3) or the Vlasov equation (B.4) with Maxwell’s equations, total charge and current density of the system (source terms); all together completes the microscopic description of the plasma.

\[
\nabla \times E = -\frac{\partial B}{\partial t} \tag{B.5}
\]

\[
\nabla \times B = \mu_0 j + \frac{1}{c^2} \frac{\partial E}{\partial t} \tag{B.6}
\]

\[
\nabla \cdot E = \frac{q_T}{\epsilon_0} \tag{B.7}
\]

\[
\nabla \cdot B = 0 \tag{B.8}
\]

where \( q_T \) and \( j \) are the charge density and density current, respectively.

In order to obtain the macroscopic variables needed for the fluid description of the plasma, which will lead to the MHD equations, a task still remains, viz. to take different moments of the Boltzmann equation (B.3) by multiplying it with powers of the velocity \( v \) and integrating over the velocity space. This will lead to the derivation of the continuity, momentum, and energy equations. However, before the latter is accomplished an intermediate step will
be developed, namely the definition of the following variables

\[ n_\alpha(r, t) \equiv \int f_\alpha(r, v, t) \, d^3v \]  

(B.9)

which tell us the number of particles of species \( \alpha \) per unit volume. Furthermore, an average velocity \( u_\alpha \) of particle \( \alpha \) will be required

\[ u_\alpha(r, t) = \langle v \rangle_\alpha = \int v f_\alpha(r, v, t) \, d^3v \]  

(B.10)

Lastly, since thermal fluctuations will take place, it is then introduced a random velocity term \( \tilde{v}_\alpha \)

\[ \tilde{v}_\alpha \equiv v - u_\alpha \]  

(B.11)

As mentioned before, moments are to be taken from the evolution equation (B.3) of the distribution function. Thus, the first equation that describes a macroscopic behaviour of the plasma is obtained when the zeroth moment of Boltzmann equation (B.3) is calculated. The latter gives rise to the continuity equation for particles of species \( \alpha \)

\[ \frac{\partial n_\alpha}{\partial t} + \nabla \cdot (n_\alpha u_\alpha) = 0 \]  

(B.12)

Secondly, the first moment of the equation (B.3) is obtained by multiplying the latter by \( u_\alpha \) and then integrating over the velocity space. Then, together with the appropriate transformations [38], it is obtained the final form of the momentum equation for particle of species \( \alpha \)

\[ n_\alpha m_\alpha \left( \frac{\partial}{\partial t} + u_\alpha \cdot \nabla \right) u_\alpha + \nabla (n_\alpha k_B T_\alpha) - n_\alpha q_\alpha (E + u_\alpha \times B) = -\nabla \cdot \pi_\alpha + R_\alpha \]  

(B.13)

where use has been made of the following definitions: the temperature \( T_\alpha \) of the \( \alpha \) species measured by the mean kinetic energy of the particle \( \alpha \), the stress tensor \( P_\alpha \) which has a traceless contribution \( \pi_\alpha \) (anisotropic contribution), and \( R_\alpha \) considered as the momentum transfer from particles \( \beta \) to particles \( \alpha \) (contribution from the collision operator \( C_\alpha = \sum_\beta C_{\alpha\beta} \)).

Last but not least, the energy equation is found when the second moment of Boltzmann equation is calculated together with the suitable substitutions

\[ \frac{3}{2} n_\alpha \left( \frac{\partial}{\partial t} + u_\alpha \cdot \nabla \right) k_B T_\alpha + n_\alpha k_B T_\alpha \nabla \cdot u_\alpha = -\nabla \cdot \pi_\alpha + \nabla \cdot h_\alpha + Q_\alpha \]  

(B.14)

note that the heat flow \( h_\alpha \) appears and \( Q_\alpha \) the heat transferred to the system of particles \( \alpha \) due to collisions with the unlike particles \( \beta \) as well.

The Eqs. (B.12)-(B.14) should be further developed if a full magnetofluid description is to be obtained. This is accomplished when \( \pi_\alpha, h_\alpha, R_\alpha, \) and \( Q_\alpha \)
are written in terms of the macroscopic variables $n_\alpha$, $u_\alpha$, and $T_\alpha$ together with the electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$. Here the relations between the transport variables and the macroscopic are only stated, the mathematically inclined readers should be aware that subtle derivations are involved and those can be found in classic papers such as [64], a more recent treatment [65, 66] and in [39, 38] the relations are presented in a condensed way. Having mentioned the latter, the closure of kinetic equations is given by

\begin{align}
\pi_\alpha &\sim \mu_\alpha \nabla u_\alpha \\
h_\alpha &\sim -\kappa_\alpha \nabla (k_B T_\alpha) \\
R_\alpha &\approx -q_\alpha n_\alpha \eta \mathbf{j}
\end{align}

\begin{align}
Q_e &= - (u_e - u_i) \cdot R_e - Q_i \\
Q_i &= \frac{3}{2} n_e k_B (T_e - T_i)
\end{align}

where the index $\alpha$ stands for either ions $i$ or electrons $e$, thermal conductivity $\kappa_\alpha$, resistivity $\eta$, and viscosity $\mu_\alpha$ depend on combinations of the characteristic relaxation time $\tau_\alpha$, gyrofrequency $\Omega_\alpha$, and temperature $T_\alpha$. As stated before, with these approximations the kinetic equations (B.12)-(B.14) reduces to the two-fluid MHD model, regarding every species $\alpha$ in the system.

Once the kinetic equations are reduced to the two fluid picture of plasma, then with some additional assumptions the single fluid MHD emerges.
Chapter 6

Bibliography


6. Bibliography


6. Bibliography
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\(^1\)Here nuclear stands for elementary family and it has nothing to do with nuclear fusion nor fission.
Declaration in lieu of oath

Herewith I declare in lieu of oath that I have prepared this thesis exclusively with the help of my scientific teachers and the means quoted by them.

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