

**Seminar paper No. 752**  
**AMBIGUITY AVERSION, THE EQUITY PREMIUM**  
**AND THE WELFARE COSTS OF BUSINESS**  
**CYCLES**

**by**  
**Irasema Alonso and Jose Mauricio Prado, Jr.**



**INSTITUTE FOR INTERNATIONAL ECONOMIC STUDIES**  
**Stockholm University**

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# Ambiguity Aversion, the Equity Premium, and the Welfare Costs of Business Cycles\*

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## Abstract

We examine the potential importance of consumer ambiguity aversion for asset prices and how consumption fluctuations influence consumer welfare. First, considering a simple Mehra-Prescott-style endowment economy with a representative agent facing consumption fluctuations calibrated to match U.S. data, we study to what extent ambiguity aversion can deliver asset prices that are consistent with data: a high return on equity and a low return on riskfree bonds. For some configurations of preference parameters—a discount factor, a degree of relative risk aversion, and a measure of ambiguity aversion—we find that it can. Then, we use these parameter configurations to investigate how much consumers would be willing to pay to reduce endowment fluctuations to zero, thus delivering a Lucas-style welfare cost of fluctuations. These costs turn out to be very large: consumers are willing to pay over 10% of consumption in permanent terms.

## 1 Introduction

We examine the potential importance of consumers' ambiguity aversion in the context of macroeconomic fluctuations: we ask how consumers price risky fluctuations and how the fluctuations influence consumer welfare. Ambiguity aversion, which is a way of formalizing preferences that are consistent with the Ellsberg paradox, captures a form of violation of Savage's axioms of subjective probability. Instead, consumers behave as if a range of probability distributions are possible and as if they are averse toward the "unknown". With the typical

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parameterized representation of ambiguity aversion, consumers have minmax preferences, thus maximizing utility based on the worst possible belief within some given set of feasible beliefs. Thus, in an economy with a small amount of randomness, there are first-order effects on utility if there is ambiguity about this randomness. Thus, ambiguity aversion is in contrast to the standard model, where risk aversion leads to second-order effects on utility.

The first step in our work is to look at asset pricing in a simple Mehra-Prescott-style endowment economy. Here, we demonstrate how larger equity premia can be obtained by assuming ambiguity aversion, along with low riskfree rates. The key parameter in the model is the amount of ambiguity aversion, but it interacts nonlinearly with other parameters, such as the coefficient of relative risk aversion. There is no direct evidence of which we are aware that allows us to calibrate the ambiguity parameter, but we show a range of calibrations that roughly match the average returns on risky and riskless assets.<sup>1</sup>

The second step of the work is to ask how consumers assess the fluctuations from a welfare point of view. Thus, we redo the Lucas (2003)-style calculation whereby it is asked by how much the representative-consumer utility would rise (expressed as a permanent increase in consumption) if all fluctuations around the trend were eliminated. The answer, in the economy with ambiguity, naturally depends on the amount of ambiguity: since ambiguity is a form of “worry” about random fluctuations, the elimination of the randomness would eliminate the worry, and consumers would be better off as a result. Here, we use asset prices as a way of calibrating the ambiguity parameter. That is, we use the first step in our work as a calibration, and then do the Lucas (2003) calculation based on it. We find the welfare costs to be of the order of magnitude of 15% of consumption. This is a huge number (Lucas found about a tenth on 1%), and it is accounted for by allowing larger risk aversion and introducing ambiguity aversion.

In assessing how ambiguity might be important in the economy, it seems relevant to consider whether there is heterogeneity in the extent to which different consumers are ambiguity-averse. The third part of our paper examines how heterogeneity in ambiguity aversion influences wealth distribution, and thus indirectly asset pricing, since consumers’ influence on prices operates through (is “weighted by”) their wealth holdings. We consider a simple case and assume that half of the agents display a given amount of ambiguity aversion while the rest (the “standard agents”) do not. We specialize to a logarithmic period utility function and *iid* and symmetric shocks. For this particular case, we are able to show that the standard agents will increasingly dominate in the pricing of the assets over time. Furthermore, with this heterogeneity, the most ambiguity-averse agents become (almost) non-participants in the stock market over time; thus, we obtain endogenous limited participation. In conclusion, although ambiguity aversion shows great potential in providing new asset-pricing implications and in allowing us to think of a reason why the elimination of aggregate fluctuations might be quite costly, heterogeneity in the degree of ambiguity aversion will

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<sup>1</sup>For a survey on the equity premium puzzle, see Kocherlakota (1996).

tend to limit these implications and mainly have effects on wealth distribution and the differences in portfolios across consumers.

## 2 The economy

This is an infinite-horizon exchange economy. Production is exogenous: the economy has a tree that pays dividends every period. The dividend grows at a random rate, which has a two-state support given by  $(\lambda_1, \lambda_2)$  and follows a first-order Markov process. The transition probabilities are given by  $\phi_{ss'}$  – the probability of going to state  $s'$  if today's state is  $s$ , with  $s, s' = 1, 2$ .

When the consumer is ambiguous about these probabilities, he perceives them to be

$$\Phi(v) = \begin{pmatrix} \phi_{11} - v_1 & \phi_{12} + v_1 \\ \phi_{21} - v_2 & \phi_{22} + v_2 \end{pmatrix}, \quad (1)$$

where  $v_s \in [-a, a]$  ( $s = 1, 2$ ) with restrictions on  $a$  such that all probabilities are in  $[0, 1]$ . Parameter  $a$  measures the amount of ambiguity in the economy.

Preferences are given by the maxmin formulation

$$V_t(s^t) = u[c(s^t)] + \beta \min_{\pi \in \Pi_{s^t}} E_{\pi} V_{t+1}(s^{t+1}), \quad (2)$$

where  $c$  is consumption,  $u(c)$  is the period utility function, and  $\Pi_{s^t}$  is a set of transition probability laws given the history  $s^t$  today.

Aversion to ambiguity is captured by the “minimization” part in the utility formulation above: the consumer behaves with pessimism, i.e., he assumes the worst possible probability distribution. For an axiomatic foundation for this preference formulation, see Gilboa and Schmeidler (1989) for the static setting and Epstein and Wang (1994) and Epstein and Schneider (2003) for a multi-period setting.

In section 3 we describe the model with a representative agent and in section 4 we look at the welfare costs of consumption variability. Finally, in section 5 we consider a model with both ambiguity-averse agents and “standard” agents who do not view the economy as ambiguous.

## 3 Representative-agent asset pricing

In this section and for simplicity, we first consider an ambiguity-averse representative agent with a logarithmic period utility function and discount factor  $\beta$ . In addition, we first assume that shocks are *iid* and symmetric, i.e.,  $\phi_{ss'} = 0.5$ . After that, we consider a CRRA period utility function and assume serially correlated shocks. Then, we calibrate the economy and report the model's performance.

There is an equity share that is competitively traded and a riskless bond that is in zero net supply. We denote the consumer's bond and equity holdings  $b$  and  $e$ , respectively.

The representative agent holds the tree and thus, his consumption in every period is the dividend of the tree. A log-period utility function together with the assumption of *iid* shocks imply that  $p$ , the price of the tree, will be linear in  $d$ , the dividend, and independent of the state:  $p(d) = \hat{p}d$ .

The ambiguity-averse consumer puts a higher weight on the bad outcome than what is warranted by the objective probability; that is, he becomes pessimistic because he is worried about that outcome and does not know its probability.

We assume that  $\lambda_1 > \lambda_2$  so that the bad outcome is state 2 – the outcome where the dividend is low. The objective probability of this state is 0.5, but he chooses the belief in the bad state. His belief is  $\phi(v) = 0.5 - v$  and he chooses  $v$  from the set  $v \in [-a, a]$ . The higher is  $a$ , the more ambiguity there is in the economy.

The problem of the representative agent with wealth today given by  $w$  is

$$V(w) = \max_e \log [w - p(d)e] + \beta \min_{v \in [-a, a]} (\phi - v)V(w'_1) + (1 - \phi + v)V(w'_2)$$

subject to

$$w'_1 = [\lambda_1 d + p(d\lambda_1)] e,$$

and

$$w'_2 = [\lambda_2 d + p(d\lambda_2)] e.$$

Here, for ease of notation, we have excluded the bond (since bond holdings must be zero in equilibrium). Moreover, the budget constraint:  $c + p(d)e + q(d)b = w$  where  $w = [d + p(d)]e_{-1} + b_{-1}$  ( $e_{-1}$  and  $b_{-1}$  are equity and bond holdings chosen in the previous period) has been substituted away. The Euler equation for equity is

$$p(d)u'(d) = \beta \{(\phi - a)[\lambda_1 d + p(\lambda_1 d)]u'(\lambda_1 d) + (1 - \phi + a)[\lambda_2 d + p(\lambda_2 d)]u'(\lambda_2 d)\}. \quad (3)$$

Clearly,  $p$  is linear in  $d$  (a constant times  $d$ ), whenever  $u'(c) = c^{-\sigma}$  (here,  $\sigma = 1$ ). Since the period utility is logarithmic, the price of equity does not depend on beliefs because the payoff and the inverse of marginal utility ( $u'$ ) are proportional to  $\lambda d$  so that the payoff times marginal utility is the same in both states. Thus,  $p(d) = \frac{\beta}{1-\beta}d$  solves the Euler equation above: the price of equity is independent of  $\phi$  and  $a$ .

Trivially here, since  $e = 1$  in equilibrium,  $w'_1 = \frac{\lambda_1 d}{1-\beta}$ ,  $w'_2 = \frac{\lambda_2 d}{1-\beta}$ , then  $V(w'_1) > V(w'_2)$ , so the solution for  $v$  is a corner, i.e.,  $v = a$ . In section 5, we show that  $v$  can be an interior solution when the economy is populated by both ambiguity-averse and standard consumers.

The Euler equation for bonds similarly gives

$$q(d)u'(d) = \beta \{(\phi - a)u'(\lambda_1 d) + (1 - \phi + a)u'(\lambda_2 d)\}. \quad (4)$$

We see that  $q$  depends on beliefs:

$$q = \beta \left[ \left( \frac{\phi}{\lambda_1} + \frac{1-\phi}{\lambda_2} \right) + a \left( \frac{1}{\lambda_2} - \frac{1}{\lambda_1} \right) \right]. \quad (5)$$

The higher is  $a$  – the more ambiguity aversion there is in the economy – the higher is the belief that the bad state will happen, and the higher is the present value of one unit tomorrow, since the probability weight placed on the state with a high marginal utility is higher.

The net expected return on equity,  $ER_e$ , is given by

$$ER_e = \frac{\phi\lambda_1 + (1-\phi)\lambda_2}{\beta} - 1, \quad (6)$$

and it is independent of the belief. The net return on bonds,  $R_b$ , decreases when ambiguity aversion increases, because  $R_b = \frac{1}{q} - 1$ .

The equity premium in this economy is

$$ER_e - R_b = \frac{\phi\lambda_1 + (1-\phi)\lambda_2}{\beta} - \frac{\lambda_1\lambda_2}{\beta[(1-\phi)\lambda_1 + \phi\lambda_2 + a(\lambda_1 - \lambda_2)]}. \quad (7)$$

If we make  $\phi = 0.5$ , then the equity premium in this economy is

$$ER_e - R_b = \frac{\lambda_1 + \lambda_2}{2\beta} - \frac{\lambda_1\lambda_2}{\beta[0.5(\lambda_1 + \lambda_2) + a(\lambda_1 - \lambda_2)]}. \quad (8)$$

When ambiguity is most extreme, i.e., when  $a = 0.5$ , the equity premium becomes

$$\frac{\lambda_1 - \lambda_2}{2\beta}.$$

Using  $\lambda_1 = 1.02$ ,  $\lambda_2 = 1.01$ , and  $\beta = 0.98$ , the equity premium is 0.5%, which is 200 times larger than the equity premium for the same parameter values when  $a = 0$  – the standard model. Although this is an example, and not a calibration, it illustrates that the effect of ambiguity on asset prices/returns can be substantial.

If the period utility is  $u(d) = \frac{d^{1-\alpha}}{1-\alpha}$ , the price of equity depends on beliefs. In fact:

$$\hat{p} = \frac{\beta [(\phi - a)\lambda_1^{1-\alpha} + (1 - \phi + a)\lambda_2^{1-\alpha}]}{1 - \beta [(\phi - a)\lambda_1^{1-\alpha} + (1 - \phi + a)\lambda_2^{1-\alpha}]}. \quad (9)$$

### 3.1 Serial correlation

We now assume that the period utility is  $u(c) = \frac{c^{1-\alpha}}{1-\alpha}$  and the shocks are serially correlated.

The problem of the representative agent with wealth today given by  $w$  and today's shock  $s$  is

$$V_s(w) = \max_e u[w - p_s(d)e] + \beta \min_{v_s \in [-a, a]} (\phi_{s1} - v_s)V_1(w'_1) + (\phi_{s2} + v_s)V_2(w'_2)$$

subject to

$$\begin{aligned} w'_1 &= [\lambda_1 d + p_1(d\lambda_1)] e, \\ w'_2 &= [\lambda_2 d + p_2(d\lambda_2)] e. \end{aligned}$$

The Euler equation for equity is

$$p_s(d)u'(d) = \beta \{(\phi_{s1} - v_s)[\lambda_1 d + p_1(\lambda_1 d)]u'(\lambda_1 d) + (\phi_{s2} + v_s)[\lambda_2 d + p_2(\lambda_2 d)]u'(\lambda_2 d)\} \quad (10)$$

The price of equity is still linear in  $d$ , and is now given by

$$p_s(d) = k_s d \quad (11)$$

where

$$k_s = \beta [(\phi_{s1} - v_s)\lambda_1^{1-\alpha}(1 + k_1) + (\phi_{s2} + v_s)\lambda_2^{1-\alpha}(1 + k_2)], \quad (12)$$

for  $s = 1, 2$ .

Explicitly solving for  $k_1$  and  $k_2$ , we obtain:

$$k_1 = \frac{\beta(\phi_{11} - a)\lambda_1^{1-\alpha} [1 - \beta(\phi_{22} + a)\lambda_2^{1-\alpha}] + \beta(\phi_{12} + a)\lambda_2^{1-\alpha} + \beta^2(\phi_{12} + a)(\phi_{21} - a)(\lambda_1\lambda_2)^{1-\alpha}}{[1 - \beta(\phi_{22} + a)\lambda_2^{1-\alpha}] [1 - \beta(\phi_{11} - a)\lambda_1^{1-\alpha}] - \beta^2(\phi_{12} + a)(\phi_{21} - a)(\lambda_1\lambda_2)^{1-\alpha}}$$

and

$$k_2 = \frac{\beta(\phi_{22} + a)\lambda_2^{1-\alpha} [1 - \beta(\phi_{11} - a)\lambda_1^{1-\alpha}] + \beta(\phi_{21} - a)\lambda_1^{1-\alpha} + \beta^2(\phi_{21} - a)(\phi_{12} + a)(\lambda_1\lambda_2)^{1-\alpha}}{[1 - \beta(\phi_{22} + a)\lambda_2^{1-\alpha}] [1 - \beta(\phi_{11} - a)\lambda_1^{1-\alpha}] - \beta^2(\phi_{12} + a)(\phi_{21} - a)(\lambda_1\lambda_2)^{1-\alpha}}$$

Thus, wealth in the next period is:

$$w'_1 = \lambda_1 d(1 + k_1), \quad (13)$$

and

$$w'_2 = \lambda_2 d(1 + k_2). \quad (14)$$

The price of the bond is given by

$$q_s(d) = \beta \left[ \phi_{s1} \frac{1}{\lambda_1^\alpha} + \phi_{s2} \frac{1}{\lambda_2^\alpha} + a \left( \frac{1}{\lambda_2^\alpha} - \frac{1}{\lambda_1^\alpha} \right) \right] \quad (15)$$

for  $s = 1, 2$ .

The conditional expected net return on equity is

$$ER_s^e = \frac{\phi_{s1} [\lambda_1 d + p_1(\lambda_1 d)] + \phi_{s2} [\lambda_2 d + p_2(\lambda_2 d)]}{p_s(d)} - 1 \quad (16)$$

for  $s = 1, 2$ , and the unconditional expected net return on equity  $ER^e$ , is

$$\pi ER_1^e + (1 - \pi)ER_2^e - 1$$

where the invariant probability  $\pi$  solves

$$\pi = \phi_{11}\pi + \phi_{21}(1 - \pi). \quad (17)$$



Therefore,

$$ER^e = \pi \frac{\phi_{11} [\lambda_1 d + p_1(\lambda_1 d)] + \phi_{12} [\lambda_2 d + p_2(\lambda_2 d)]}{p_1(d)} + (1 - \pi) \frac{\phi_{21} [\lambda_1 d + p_1(\lambda_1 d)] + \phi_{22} [\lambda_2 d + p_2(\lambda_2 d)]}{p_2(d)} - 1 \quad (18)$$

$$ER^e = \pi \frac{\phi_{11} \lambda_1 (1 + k_1) + \phi_{12} \lambda_2 (1 + k_2)}{k_1} + (1 - \pi) \frac{\phi_{21} \lambda_1 (1 + k_1) + \phi_{22} \lambda_2 (1 + k_2)}{k_2} - 1 \quad (19)$$

The expected net return on the bond,  $R^b$ , is given by

$$\pi \frac{1}{q_1} + (1 - \pi) \frac{1}{q_2} - 1 = \frac{1}{\beta} \left[ \frac{\pi}{\phi_{11} \frac{1}{\lambda_1^\alpha} + \phi_{12} \frac{1}{\lambda_2^\alpha} + a \left( \frac{1}{\lambda_2^\alpha} - \frac{1}{\lambda_1^\alpha} \right)} + \frac{(1 - \pi)}{\phi_{21} \frac{1}{\lambda_1^\alpha} + \phi_{22} \frac{1}{\lambda_2^\alpha} + a \left( \frac{1}{\lambda_2^\alpha} - \frac{1}{\lambda_1^\alpha} \right)} \right] - 1. \quad (20)$$

Finally, the equity premium is given by

$$ER^e - R^b.$$

### 3.2 Calibration and evaluation of asset prices

As in Mehra and Prescott (1985), we now select the parameters of the model so that the average growth rate of per capita consumption, the standard deviation of the growth rate of per capita consumption and the first-order serial correlation of this growth rate, all with respect to the model's stationary distribution, match the sample values for the U.S. economy between 1889-1978.

The values of the parameters are  $\phi = 0.43$  (where  $\phi_{11} = \phi_{22} = \phi$  and  $\phi_{12} = \phi_{21} = (1 - \phi)$ ),  $\delta = 0.036$ ,  $\mu = 0.018$ ,  $\lambda_1 = 1 + \mu + \delta = 1.054$ , and  $\lambda_2 = 1 + \mu - \delta = 0.982$ .

Figure 1 shows the return on the risk-free bond, the expected return on equity and the equity premium for  $\beta = 0.95$ ,  $a = 0.2$ , and for a range of  $\alpha$  between 0 and 10.

The equity premium is higher as  $\alpha$  increases. Note, for example, that for  $\alpha = 8$ , the risk-free return is 4.72%, the expected return on equity is 8.77%, and the resulting equity premium is 4.05%.

Figure 2 shows the return on the risk-free asset, the expected return on equity, and the equity premium for  $\beta = 0.95$ ,  $\alpha = 2$ , and the ambiguity parameter  $a$  in a range between 0 and 0.43.

The equity premium increases with the amount of ambiguity in the economy. For example, for  $a = 0.3$ , the return on the bond is 4.27%, the expected return on equity is 6.98%, and the resulting equity premium is 2.71%. As a comparison, the largest equity premium that Mehra and Prescott (1985) were able to obtain was 0.35%.

## 4 Potential benefits of eliminating consumption fluctuations

We first calculate the costs of consumption fluctuations when shocks are *iid*. The present discounted utility when the dividend today is  $d$  is given recursively by

$$V(d) = \frac{d^{1-\alpha}}{1-\alpha} + \beta \left\{ \min_{v \in [-a, a]} [\phi V(\lambda_1 d) + (1-\phi)V(\lambda_2 d)] \right\}. \quad (21)$$

The solution for  $V(d)$  is

$$V(d) = Ad^{1-\alpha}, \quad (22)$$

where

$$A = \frac{1}{(1-\alpha) \{1 - \beta [(\phi - a)\lambda_1^{1-\alpha} + (1-\phi + a)\lambda_2^{1-\alpha}]\}}. \quad (23)$$

Moreover,  $v = a$  since  $V(d)$  is increasing in  $d$ .

Eliminating consumption fluctuations will deliver the present value of total utility corresponding to consuming the expected value of the dividend every period. This utility is given by:

$$\sum_{t=0}^{\infty} \beta^t \frac{\left\{ d [\phi \lambda_1 + (1-\phi)\lambda_2]^t \right\}^{1-\alpha}}{1-\alpha} = \frac{d^{1-\alpha}}{(1-\alpha) \left\{ 1 - \beta [\phi \lambda_1 + (1-\phi)\lambda_2]^{1-\alpha} \right\}}. \quad (24)$$

The costs of consumption variability are given by  $\gamma$  where  $\gamma$  solves:

$$\frac{(1-\gamma)^{1-\alpha}}{1-\beta [\phi \lambda_1 + (1-\phi)\lambda_2]^{1-\alpha}} = \frac{1}{1-\beta [(\phi - a)\lambda_1^{1-\alpha} + (1-\phi + a)\lambda_2^{1-\alpha}]}. \quad (25)$$

Calculating the utility of the deterministic growth path is more evolving when the shocks are serially correlated. To this end, we will now introduce some notation.

Let the transition probabilities be given by

$$\Phi \equiv \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix}, \quad (26)$$

let

$$\Lambda \equiv \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad (27)$$

and let

$$\lambda \equiv \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}. \quad (28)$$

Consider the expression

$$\lambda_t^e \equiv (\Phi \Lambda)^{t-1} \Phi \lambda. \quad (29)$$

The first row of this expression,  $\lambda_t^e$ , is the *expected growth rate between now and  $t$  periods from now if the state now is state 1*; and the second row,  $\lambda_t^e$ , is the *expected growth rate between now and  $t$  periods from now if the state now is state 2*. Denote these  $\lambda_t^e|_1$  and  $\lambda_t^e|_2$ , respectively; that is,

$$\lambda_t^e \equiv \begin{pmatrix} \lambda_t^e|_1 \\ \lambda_t^e|_2 \end{pmatrix}. \quad (30)$$

The utility of the deterministic growth path, where growth is equal to the expected value beginning in state  $s$ , is

$$u(d) + \beta u(d\lambda_1^e|_s) + \beta^2 u(d\lambda_2^e|_s) + \beta^3 u(d\lambda_3^e|_s) + \dots,$$

which when we have CRRA utility equals

$$\frac{d^{1-\alpha}}{1-\alpha} \left[ 1 + \beta (\lambda_1^e|_s)^{1-\alpha} + \beta^2 (\lambda_2^e|_s)^{1-\alpha} + \beta^3 (\lambda_3^e|_s)^{1-\alpha} + \dots \right]. \quad (31)$$

The present value of total utility when the dividend is  $d$  and the shock is  $s$ , is given by

$$V_s(d) = A_s d^{1-\alpha} \quad (32)$$

for  $s = 1, 2$ , and where

$$A_1 = \frac{1 + \beta \lambda_2^{1-\alpha} (\phi_{12} - \phi_{22})}{(1-\alpha) \left\{ [1 - \beta(\phi_{22} + a)\lambda_2^{1-\alpha}] [1 - \beta(\phi_{11} - a)\lambda_1^{1-\alpha}] - \beta^2 (\phi_{21} - a)(\phi_{12} + a)(\lambda_1 \lambda_2)^{1-\alpha} \right\}},$$

and

$$A_2 = \frac{1 + \beta \lambda_1^{1-\alpha} (\phi_{21} - \phi_{11})}{(1-\alpha) \left\{ [1 - \beta(\phi_{22} + a)\lambda_2^{1-\alpha}] [1 - \beta(\phi_{11} - a)\lambda_1^{1-\alpha}] - \beta^2 (\phi_{21} - a)(\phi_{12} + a)(\lambda_1 \lambda_2)^{1-\alpha} \right\}}.$$

Thus, the welfare cost starting from state 1 is given by the  $\gamma_1$  solving

$$A_1 = \frac{(1 - \gamma_1)^{1-\alpha}}{1 - \alpha} \left[ 1 + \beta (\lambda_1^e|_1)^{1-\alpha} + \beta^2 (\lambda_2^e|_1)^{1-\alpha} + \beta^3 (\lambda_3^e|_1)^{1-\alpha} + \dots \right]. \quad (33)$$

Similarly, the welfare cost starting from state 2 is given by the  $\gamma_2$  solving

$$A_2 = \frac{(1 - \gamma_2)^{1-\alpha}}{1 - \alpha} \left[ 1 + \beta (\lambda_1^e|_2)^{1-\alpha} + \beta^2 (\lambda_2^e|_2)^{1-\alpha} + \beta^3 (\lambda_3^e|_2)^{1-\alpha} + \dots \right]. \quad (34)$$

Figure 3 plots the costs of business cycles for  $\beta = 0.9$ , and  $\alpha = 2$  as a function of  $a$ ; i.e., it shows a “comparative-statics” exercise with respect to the ambiguity parameter only.

Clearly, more ambiguity aversion increases the costs of business cycles. By eliminating fluctuations (if that is possible), the government would eliminate the first-order negative effect on utility that consumers experience from random consumption.

We continue with comparative statics with respect to various parameters and then finally describe the welfare costs when the parameters are selected to match the asset prices.

Figure 4 shows the costs of business cycles for  $\beta = 0.7$ , and  $a = 0.1$  as a function of  $\alpha$ . Consumption fluctuations hurt more the more risk averse is a consumer. However, this result is not true for very high values of  $\beta$  or very high values of  $a$ .

Finally, figure 5 plots the costs of business cycles for  $\alpha = 2$  and  $a = 0.2$  as a function of  $\beta$ .

We now look at the costs of fluctuations when the asset prices match the data. As was discussed briefly above, this can be accomplished in different ways, and each of these is associated with a different cost. Table 1 illustrates that the welfare costs—or, rather, the *potential* welfare costs—of cycles are huge. They do not differ markedly across the different parameter configurations.

Table 1: Costs of business cycles for selected parameters and  $a > 0$

$\beta$	$\alpha$	$a$	$ER^e - R^b$	$\gamma_1$	$\gamma_2$
0.95	13.74	0.2040	6.18%	12.48%	12.46%
0.94	13.36	0.2223	6.18%	12.88%	12.86%
0.93	12.95	0.2420	6.18%	13.32%	13.30%
0.92	12.46	0.2642	6.18%	13.86%	13.85%
0.91	11.98	0.2879	6.18%	14.43%	14.41%
0.90	11.37	0.3160	6.18%	15.19%	15.17%
0.89	10.70	0.3480	6.18%	16.06%	16.05%
0.88	9.83	0.3890	6.18%	17.33%	17.32%
0.873	8.94	0.4300	6.18%	18.76%	18.75%

Finally, for comparison, we show the associated costs for  $a = 0$ . These are also high compared to Lucas' (2003) numbers, since  $\alpha$  is high, but of an order of magnitude lower than above.

Table 2: Costs of business cycles for selected parameters and  $a = 0$

$\beta$	$\alpha$	$a$	$R^b$	$ER^e - R^b$	$\gamma_1$	$\gamma_2$
0.95	13.74	0	19.49%	4.40%	2.55%	2.53%
0.94	13.36	0	20.67%	4.25%	2.41%	2.39%
0.93	12.95	0	21.85%	4.08%	2.28%	2.26%
0.92	12.46	0	23.01%	3.88%	2.15%	2.13%
0.91	11.98	0	24.16%	3.68%	2.02%	2.01%
0.90	11.37	0	25.25%	3.43%	1.89%	1.88%
0.89	10.70	0	26.27%	3.15%	1.76%	1.75%
0.88	9.83	0	27.12%	2.80%	1.62%	1.61%
0.873	8.94	0	27.42%	2.45%	1.50%	1.49%

## 5 Heterogeneity in ambiguity aversion

We now consider two types of agents whose ambiguity aversions differ. We look at a general planning problem first, and then focus on the case with *iid* shocks. Later, we look at the case of serial correlation in more detail.

### 5.1 The planner's problem

The state vector is  $(d, \theta, s)$ : today's dividend, the weight the planner puts on consumer 1, and today's shock. The planner solves the problem

$$V_s(d, \theta) = \max_{c_1, c_2, z_{1s'}, z_{2s'}} \theta \log c_1 + (1 - \theta) \log c_2 +$$

$$+\beta \left\{ \min_{v_1 \in [-a_1, a_1]} \theta \sum_{s'=1}^2 \phi_{ss'}(v_1) z_{1s'} + \min_{v_2 \in [-a_2, a_2]} (1 - \theta) \sum_{s'=1}^2 \phi_{ss'}(v_2) z_{2s'} \right\}$$

subject to

$$\min_{\theta'_{s'}} V_{s'}(d\lambda_{s'}, \theta'_{s'}) - [\theta'_{s'} z_{1s'} + (1 - \theta'_{s'}) z_{2s'}] \geq 0, \quad (35)$$

and

$$c_1 + c_2 = d, \quad (36)$$

where  $c_i$  is agent  $i$ 's consumption,  $i = 1, 2$ ,  $z_i$  is next period's present-value utility for agent  $i$ ,  $\phi_{s1}(v_i) = \phi_{s1} - v_i$ , and  $\phi_{s2}(v_i) = \phi_{s2} + v_i$ . The first constraint (35) makes the problem recursive and the second constraint (36) is the resource constraint. This formulation which is based on Lucas and Stokey (1984) is also used in Alonso (2007).

Taking FOCs with respect to the consumption of agents 1 and 2, we have

$$c_1 = \theta d, \quad (37)$$

and

$$c_2 = (1 - \theta)d, \quad (38)$$

with respect to  $z_1(1)$  and  $z_2(1)$ , we obtain

$$\frac{\theta'_1}{1 - \theta'_1} = \frac{\theta(\phi_{s1} - v_1)}{(1 - \theta)(\phi_{s1} - v_2)}, \quad (39)$$

and similarly with respect to  $z_1(2)$  and  $z_2(2)$  we have

$$\frac{\theta'_2}{1 - \theta'_2} = \frac{\theta(\phi_{s2} + v_1)}{(1 - \theta)(\phi_{s2} + v_2)}. \quad (40)$$

After some algebra, we can rewrite the planner's problem as

$$V_s(d, \theta) = \max_{c_1, c_2} \theta \log c_1 + (1 - \theta) \log c_2 +$$

$$+\beta \min_{v_1, v_2} \left\{ \sum_{s'=1}^2 \phi_{ss'} [\theta v_1 + (1-\theta)v_2] V_{s'}(d\lambda_{s'}, \theta'_{s'}) \right\}$$

subject to

$$\theta'_{s'} = \theta \frac{\phi_{ss'}(v_1)}{\phi_{ss'}[\theta v_1 + (1-\theta)v_2]}, \quad (41)$$

and

$$c_1 + c_2 = d. \quad (42)$$

Note that  $\phi_{s1}[\theta v_1 + (1-\theta)v_2] = \phi_{s1} - \theta v_1 - (1-\theta)v_2$  and  $\phi_{s2}[\theta v_1 + (1-\theta)v_2] = \phi_{s2} + \theta v_1 + (1-\theta)v_2$ .

### 5.1.1 A special case: no serial correlation and $a_2 = 0$

In the simpler case where shocks are *iid* and symmetric and consumer 2 is not ambiguity-averse ( $v_2 = a_2 = 0$ ), the planner's problem becomes<sup>2</sup>:

$$V(d, \theta) = \max_{c_1, c_2, \theta'_{s'}} \theta \log c_1 + (1-\theta) \log c_2 +$$

$$+\beta \min_{v \in [-a, a]} \left\{ \sum_{s'=1}^2 \phi_{s'}(\theta v) V(d\lambda_{s'}, \theta'_{s'}) \right\}$$

subject to

$$\theta'_{s'} = \theta \frac{\phi_{s'}(v)}{\phi_{s'}(\theta v)}, \quad (43)$$

and

$$c_1 + c_2 = d. \quad (44)$$

Using the FOCs for consumption, we obtain  $c_1 = \theta d$  and  $c_2 = (1-\theta)d$ , so we get

$$V(d, \theta) = \log d + \log \theta^\theta (1-\theta)^{1-\theta} +$$

$$+\beta \min_v [(\phi - \theta v)V(d\lambda_1, \theta'_1) + (1 - \phi + \theta v)V(d\lambda_2, \theta'_2)]$$

with

$$\theta'_1 = \theta \frac{\phi - v}{\phi - \theta v}, \quad (45)$$

and

$$\theta'_2 = \theta \frac{1 - \phi + v}{1 - \phi + \theta v}. \quad (46)$$

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<sup>2</sup>From now on, we drop the subscript on  $v$  and  $a$ , since it should be clear that they refer only to consumer 1.

Here, we conjecture that  $V(d, \theta)$  takes the form  $A \log d + W(\theta)$ . This guess delivers

$$A \log d + W(\theta) = \log d + \log \theta^\theta (1 - \theta)^{1-\theta} + \\ + \beta \min_v \left\{ (\phi - \theta v) [A \log(d\lambda_1) + W(\theta'_1)] + (1 - \phi + \theta v) [A \log(d\lambda_2) + W(\theta'_2)] \right\}.$$

Inspecting this functional equation, it can be seen that  $A = \frac{1}{1-\beta}$  works and we can express  $W(\theta)$  as

$$W(\theta) = \log \theta^\theta (1 - \theta)^{1-\theta} + \\ + \beta \min_{v \in [-a, a]} \left\{ (\phi - \theta v) \left[ \frac{\log \lambda_1}{1 - \beta} + W \left( \theta \frac{\phi - v}{\phi - \theta v} \right) \right] + (1 - \phi + \theta v) \left[ \frac{\log \lambda_2}{1 - \beta} + W \left( \theta \frac{1 - \phi + v}{1 - \phi + \theta v} \right) \right] \right\}.$$

This is a one-dimensional dynamic programming problem delivering optimal  $v$  as a function of  $\theta$  and hence, a law of motion for  $\theta$ . The variable  $\theta$  also corresponds to the fraction of the total wealth—the current dividend plus the value of the tree—owned by agent 1 in a complete-markets equilibrium. The following figures for  $W(\theta)$  and  $v(\theta)$  below assume the same values for the parameters as specified at the end of section 3.

Figure 6, for  $W(\theta)$ , reveals a shape similar to  $\log \theta^\theta (1 - \theta)^{1-\theta}$ , which is the (constant) flow utility of a planner in a two-type economy where no consumer has ambiguity aversion.

Figure 7, for the optimal choice of  $v$ , shows that  $v$  is close to zero and interior at first (for small  $\theta$ 's), and then it increases monotonically in  $\theta$  and reaches the upper bound  $a$  for a value of  $\theta$  a little above 0.9. We will interpret these findings in more detail in the following sections.

## 5.2 The special (*iid*) case: the decentralized economy

Markets are complete and consumers trade in equity shares of the tree and in a riskless bond. The consumer's problem is given recursively by

$$V(d, w, \theta) = \max_{c, b, e} \left\{ \log c + \beta \min_v \sum_{s'=1}^2 \phi_{s'}(v) V(\lambda_{s'} d, w'_{s'}, \theta'_{s'}) \right\},$$

subject to the budget constraint

$$c + p(d, \theta)e + q(d, \theta)b = w, \quad (47)$$

$$w'_{s'} = b + e [\lambda_{s'} d + p(\lambda_{s'} d, \theta'_{s'})], \quad (48)$$

and the law of motion for  $\theta'_{s'}$  given by

$$\theta'_{s'} = g_{s'}(d, \theta), \quad (49)$$

where  $(d, w, \theta)$  is the state vector. As before,  $w$  is the consumer's wealth today,  $p$  is the price of equity,  $e$  is the fraction of the equity share held by the consumer,  $q$  is the price today of a bond that pays one unit of the consumption good next period, and  $b$  is the holdings of the bond. (The argument  $d$  is included for  $g$  only for completeness; it will not be there under the log assumption.)

The consumers' decision rules for all  $(d, w, \theta)$  are

$$c_i(d, w, \theta) \quad (50)$$

$$b_i(d, w, \theta) \quad (51)$$

$$e_i(d, w, \theta) \quad (52)$$

for  $i \in \{1, 2\}$ .

Total wealth in the economy when the state variable is  $(d, \theta)$  is  $d + p(d, \theta)$ . Thus, market clearing requires, for all values of the arguments,

$$c_1(d, \theta [d + p(d, \theta)], \theta) + c_2(d, (1 - \theta) [d + p(d, \theta)], \theta) = d, \quad (53)$$

$$b_1(d, \theta [d + p(d, \theta)], \theta) + b_2(d, (1 - \theta) [d + p(d, \theta)], \theta) = 0, \quad (54)$$

and

$$e_1(d, \theta [d + p(d, \theta)], \theta) + e_2(d, (1 - \theta) [d + p(d, \theta)], \theta) = 1. \quad (55)$$

The relative wealth dynamics, finally, is given by

$$g_{s'}(d, \theta) = \frac{w'_{1s'}(d, \theta)}{w'_{1s'}(d, \theta) + w'_{2s'}(d, \theta)}, \quad (56)$$

where

$$w'_{1s'}(d, \theta) \equiv b_1(d, \theta [d + p(d, \theta)], \theta) + e_1(d, \theta [d + p(d, \theta)], \theta) (d \lambda_{s'} + p [d \lambda_{s'}, g_{s'}(d, \theta)]),$$



and

$$w'_{2s'}(d, \theta) \equiv b_2(d, (1-\theta)[d + p(d, \theta)], \theta) + e_2(d, (1-\theta)[d + p(d, \theta)], \theta)(d\lambda_{s'} + p[d\lambda_{s'}, g_{s'}(d, \theta)]).$$

Now we will show how to find prices and portfolio allocations in this economy. We use the planning problem and we identify the  $\theta$  in that problem with the corresponding variable here: the planning weight on agent 1 equals the relative fraction of total wealth held in equilibrium by this agent.

The price of bonds,  $q(d, \theta)$ , then becomes

$$q(d, \theta) = \beta \left[ \frac{\phi}{\lambda_1} + \frac{1-\phi}{\lambda_2} + \theta v \left( \frac{1}{\lambda_2} - \frac{1}{\lambda_1} \right) \right] \equiv \hat{q}(\theta). \quad (57)$$

As shown in section 3, the price of the bond is increasing in  $a$ . In addition here, the price of bonds is increasing in  $\theta$ . We show below that the ambiguity-averse agents demand the bond. The bond is more valuable when marginal utility of consumption is high (which occurs in the bad state). As  $\theta$  increases, there is a higher demand for the bond, so its price goes up.

And the price of equity,  $p(d, \theta)$ , is given by

$$p(d, \theta) = \beta \left\{ \frac{(\phi - \theta v) [\lambda_1 d + p(\lambda_1 d, \theta'_1)]}{\lambda_1} + \frac{(1 - \phi + \theta v) [\lambda_2 d + p(\lambda_2 d, \theta'_2)]}{\lambda_2} \right\}, \quad (58)$$

where we recall that

$$\theta'_1 = \frac{\theta(\phi - v)}{\phi - \theta v} \leq \theta, \quad (59)$$

and

$$\theta'_2 = \frac{\theta(1 - \phi + v)}{1 - \phi + \theta v} \geq \theta, \quad (60)$$

from the planning problem. (The inequalities above follow since  $v \geq 0$ .)

The latter laws of motion reveal that the ambiguity-averse agent gains in relative wealth when the state is bad and loses when it is good: his probability “beliefs” are tilted toward the bad state.

We see that  $p(d, \theta) = d\hat{p}(\theta)$  solves this equation, delivering

$$\hat{p}(\theta) = \beta \{ (\phi - \theta v) [1 + \hat{p}(\theta'_1)] + (1 - \phi + \theta v) [1 + \hat{p}(\theta'_2)] \}. \quad (61)$$

This is a functional equation: it holds for all  $\theta$  (recall that  $v$  may also depend on  $\theta$ ). The solution to this functional equation is

$$\hat{p}(\theta) = \frac{\beta}{1 - \beta}, \quad (62)$$

and

$$p(d, \theta) = d \frac{\beta}{1 - \beta}. \quad (63)$$

So the price of equity does not depend on  $\theta$ .

The equilibrium holdings of equity of consumer 1, which can be obtained by using the expression for future wealth,  $w'_{1s'} = b_1 + e_1(\lambda_{s'}d + p'_{s'})$ , together with the equilibrium condition that  $w'_{1s'} = \theta'_{s'}(d\lambda_{s'} + p'_{s'})$ , are given by

$$e_1(d, \theta) = \frac{\theta'_1 \lambda_1 - \theta'_2 \lambda_2}{\lambda_1 - \lambda_2} \equiv \hat{e}_1(\theta). \quad (64)$$

Thus, the equity holdings of agent 1 are independent of the level of  $d$ . We see that if  $v = 0$ , in which case  $\theta'_1 = \theta'_2 = \theta$ , then  $\hat{e}_1(\theta) = \theta$ : the consumer's share of the tree equals his initial share of total wealth.

On the other hand, when  $v > 0$  (recall that wlog we use  $\lambda_1 > \lambda_2$ ), we know that  $\theta'_1 < \theta < \theta'_2$ , which makes the holdings of equity lower as compared to the case when  $v = 0$ . That is, the ambiguity-averse agent will have a smaller share of equity holdings than his overall wealth would otherwise prescribe: this is a *portfolio composition* effect. How much his portfolio composition will be changed must be numerically examined.

We can also examine the portfolio effect by looking at the amount of bonds purchased by agent 1. Her equilibrium holdings of bonds are obtained as

$$b_1(d, \theta) = \frac{d\lambda_1}{1 - \beta} [\theta'_1 - \hat{e}_1(\theta)]. \quad (65)$$

It is interesting to note here that bond holdings are proportional to  $d$ . Naturally, they are zero in the special case  $v = 0$ , when  $e_1 = \theta$  and  $\theta' = \theta$ . Moreover,

$$\begin{aligned} \theta'_1 - \hat{e}_1(\theta) &= \theta'_1 - \frac{\theta'_1 \lambda_1 - \theta'_2 \lambda_2}{\lambda_1 - \lambda_2} = \\ &= \theta'_1 \left( 1 - \frac{\lambda_1 - \frac{\theta'_2}{\theta'_1} \lambda_2}{\lambda_1 - \lambda_2} \right) > 0, \end{aligned} \quad (66)$$

since  $\theta'_2 > \theta'_1$ , and thus we conclude, consistently with the above insights regarding equity holdings, that the ambiguity-averse agent *increases* his bond holdings relative to the  $v = 0$  zero-bonds case: his portfolio composition moves away from equity and into bonds because he is more pessimistic than person 2 in his perception of the return (performance) of equity.

There are two sources of uncertainty in this economy: (i) the payoff of equity and (ii) the price of the bond. The price of the bond depends on  $\theta$ , the relative wealth of consumer 1, and this variable is random. In particular, since  $\theta'_2 > \theta > \theta'_1$ , the price of the bond,  $q$ , increases if state 2 occurs and it decreases if state 1 occurs.

Below we numerically compute solutions for  $v(\theta)$ ,  $\theta'_1(\theta)$ ,  $\theta'_2(\theta)$ ,  $e(\theta)$ ,  $\frac{p(\theta)e(\theta)}{p(\theta)e(\theta)+q(\theta)b(\theta)}$ ,  $q(\theta)$ , and  $b(\theta)$  for agent 1. Once more, the parameter values are  $\lambda_1 = 1.02$ ,  $\lambda_2 = 1.01$ , and  $\beta = 0.98$ .

As we see from the graphs in figure 8, the ambiguity-averse consumer short-sells equity for most values of  $\theta$ . The reason for this is the following. State 2 is bad for the ambiguity-averse consumer for two reasons: (i) the payoff from equity

is low and (ii) the price of the bond increases so that it makes the good next period more expensive (this consumer does not own any goods next period). Therefore, to provide protection against the former type of uncertainty, the ambiguity-averse consumer buys bonds and to provide protection against the latter type of uncertainty, the ambiguity-averse consumer sells equity short.

The behavior of the ambiguity-averse consumer can be separately described for different ranges of  $\theta$ . First, when  $\theta$  is zero, the ambiguity-averse consumers have zero aggregate wealth. In this case, the price of bonds is solely determined by the “standard” agents and it does not fluctuate. Since there is no uncertainty on  $q$ , ambiguity-averse consumers only hold bonds. As shown in the below section, a very small amount of pessimism rationalizes this choice.

If  $\theta$  is positive but small, changes in  $\theta$  do not have any considerable effects on  $q$ , so the randomness in  $q$  is not so important. Then, ambiguity-averse consumers mainly hold bonds and short-sell equity somewhat to protect against the uncertainty in  $p$ . This asset choice makes

$$V(w'_1, \theta'_1) = V(w'_2, \theta'_2) \quad (67)$$

for a small value of the belief  $v$ ; that is,  $v$  is still an interior solution.

When  $\theta$  is high, ambiguity aversion makes the fluctuations in  $q$  very large. Agents buy bonds and short-sell equity more heavily. The value of  $v$  is larger, reflecting more pessimism about state 2. Since  $V$  is decreasing in  $\theta$  and increasing in  $w$  (the former is true because  $q$  is increasing in  $\theta$ ), and since  $\theta'_1$  is much larger than  $\theta'_2$ ,  $w'_2$  needs to be much larger than  $w'_1$  in order to equate  $V(w'_1, \theta'_1)$  and  $V(w'_2, \theta'_2)$  – and hence still make  $v$  an interior solution. This is achieved by short-selling equity even more heavily.

When  $\theta$  is very close to 1,  $v$  is a corner solution since the ambiguity-averse agents need to hold most of the stock and they are pessimistic about state 1. The fluctuations in  $\theta$  have become very small, and the uncertainty resulting from changes in  $q$  is therefore also very small and ambiguity-averse agents consequently do not need to short-sell the stock.

### 5.3 Relative consumption and wealth in the long run

We can analytically show<sup>3</sup> that

$$E(\theta'|\theta) < \theta, \quad (68)$$

i.e., that over time, the relative wealth of the ambiguity-averse agents decreases toward zero: these agents disappear, economically speaking.

However, it can also be shown that

$$E\left(\frac{\theta'}{\theta}\right) \rightarrow_{\theta \rightarrow 0} 1, \quad (69)$$

so the rate at which they disappear goes to zero: they remain with positive wealth for a long, long time.

<sup>3</sup>The proofs of expressions (68) and (69) are in the appendix. This result and the following discussion are reminiscent of the analysis in Coen-Pirani (2004).

## 6 Conclusion

In this essay, we have studied asset pricing and evaluated the welfare costs of fluctuations in consumption for an economy where consumers are ambiguity-averse. First, we have shown parameter configurations under which the equity premium is quite large (and the riskfree rate is small); the ability to match these return features comes from the ability of ambiguity aversion to generate *first-order* effects on prices, which sets it apart from risk aversion, which operates through second-order effects. Ambiguity aversion has first-order effects, in essence, because consumers behave as if they believed that the good return outcomes to be less likely than they really are.

Second, using the calibrations that deliver realistic asset prices, we have shown that the welfare benefits of eliminating consumption fluctuations need not be as small as those in Lucas’s (2003) calculations. This is not to say that the benefits are large: the numbers we obtain are, just like Lucas’s numbers, upper bounds, and these upper bounds leave open what the costs of stabilization (say, in the form of distortions) might be, and also leave open whether full stabilization is even feasible. Nevertheless, it is valuable to note that these bounds can be as large as 15% of consumption when asset prices are matched by the model.

Third, by exploring an economy where some consumers are ambiguity-averse and others are not, we find an important qualification to the above findings: it appears that, by making consistently “bad bets”, ambiguity-averse consumers will see their relative wealth decline over time, and thereby asset prices will be increasingly dominated by standard consumers. Note also that these bad bets are not bad in the sense of “crazy portfolios”, but simply in the sense of delivering a lower return on average by not investing enough in stock. In particular, if ambiguity aversion is sufficiently large, the ambiguity-averse consumers choose to not participate at all in the stock market: the other, standard consumers hold all risk (and get all the high returns on average).<sup>4</sup> To make this wealth distribution not converge to an extreme outcome, one could consider an overlapping-generations structure, where in each generation of newborns with zero debt, some are ambiguity-averse; that way, a significant part of aggregate wealth will always belong to ambiguity-averse consumers.

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<sup>4</sup>It is interesting to note that there is (close to) non-participation for a large range of values for  $\theta$ . Thus, without having to assume that there are costs of transacting/investing in stock, we can use this setting to derive conditions under which a large fraction of the population—the ambiguity-averse—(almost) do not have any stock. This kind of result was also derived in Epstein and Schneider’s (2007) work. Exact non-participation cannot be obtained here because the risk-free rate fluctuates with the endowment shock; because the ambiguity-averse agents hold bonds, it is optimal for them to use equity to hedge against the interest-rate risk. This risk, however, is very small for a large range of (low) values of  $\theta$ : when  $\theta$  is zero, the risk-free rate is constant, and thus not until the ambiguity-averse agents have a significant fraction of total wealth will these fluctuations be large enough to induce significant equity holdings for these agents.

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## Appendix

### A1 Heterogeneity in ambiguity aversion

#### 6.0.1 The planning problem

We can restate the problem:

$$V_s(d, \theta) = \max_{c_1, c_2} \theta \log c_1 + (1 - \theta) \log c_2 +$$

$$+ \beta \min_{v_1, v_2} \left\{ \sum_{s'=1}^2 \phi_{ss'} [\theta v_1 + (1 - \theta)v_2] V_{s'}(d\lambda_{s'}, \theta'_{s'}) \right\}$$

subject to

$$\theta'_{s'} = \theta \frac{\phi_{ss'}(v_1)}{\phi_{ss'}[\theta v_1 + (1 - \theta)v_2]}, \quad (70)$$

and

$$c_1 + c_2 = d. \quad (71)$$

Taking FOCs, we obtain  $c_1 = \theta d$  and  $c_2 = (1 - \theta)d$ . The rewritten problem becomes:

$$V_s(d, \theta) = \log d + \log \theta^\theta (1 - \theta)^{1-\theta} +$$

$$+ \beta \min_{v_1, v_2} \{ [\phi_{s1} - \theta v_1 - (1 - \theta)v_2] V_1(d\lambda_1, \theta'_1) + [\phi_{s2} + \theta v_1 + (1 - \theta)v_2] V_2(d\lambda_2, \theta'_2) \}$$

subject to

$$\theta'_{s'} = \theta \frac{\phi_{ss'}(v_1)}{\phi_{ss'}[\theta v_1 + (1 - \theta)v_2]}. \quad (72)$$

We conjecture that  $V_s(d, \theta)$  takes the form  $A \log d + W_s(\theta)$ . This guess delivers

$$A \log d + W_s(\theta) = \log d + \log \theta^\theta (1 - \theta)^{1-\theta} +$$

$$+ \beta \min_{v_1, v_2} \{ [\phi_{s1} - \theta v_1 - (1 - \theta)v_2] [A \log d\lambda_1 + W_1(\theta'_1)] + [\phi_{s2} + \theta v_1 + (1 - \theta)v_2] V_2(d\lambda_2, \theta'_2) \}$$

#### Special case: serial correlation and $a_2 = 0$

In this case, we have

$$V_s(d, \theta) = \max_{c_1, c_2, \theta'_{s'}} \theta \log c_1 + (1 - \theta) \log c_2 +$$

$$+ \beta \min_{v_s} \left\{ \sum_{s'=1}^2 \phi_{ss'}(\theta v) V_{s'}(d\lambda_{s'}, \theta'_{s'}) \right\}$$

subject to

$$\theta'_{s'} = \theta \frac{\phi_{ss'}(v_s)}{\phi_{ss'}(\theta v_s)}, \quad (73)$$

and

$$c_1 + c_2 = d. \quad (74)$$

Using the FOCs for consumption, we obtain  $c_1 = \theta d$  and  $c_2 = (1 - \theta)d$  so we get

$$V_s(d, \theta) = \log d + \log \theta^\theta (1 - \theta)^{1-\theta} + \\ + \beta \min_{v_s} [(\phi_{s1} - \theta v_s) V_1(d\lambda_1, \theta'_1) + (\phi_{s2} + \theta v_s) V_2(d\lambda_2, \theta'_2)]$$

with

$$\theta'_1 = \theta \frac{\phi_{s1} - v_s}{\phi_{s1} - \theta v_s}, \quad (75)$$

and

$$\theta'_2 = \theta \frac{\phi_{s2} + v_s}{\phi_{s2} + \theta v_s}. \quad (76)$$

Here, we conjecture that  $V_s(d, \theta)$  takes the form  $A \log d + W_s(\theta)$ . This guess delivers

$$A \log d + W_s(\theta) = \log d + \log \theta^\theta (1 - \theta)^{1-\theta} + \\ + \beta \min_{v_s} \left( (\phi_{s1} - \theta v_s) (A \log(d\lambda_1) + W_1(\theta'_1)) + (\phi_{s2} + \theta v_s) (A \log(d\lambda_2) + W_2(\theta'_2)) \right).$$

Inspecting the above expression, it can be seen that  $A = \frac{1}{1-\beta}$  works and leaves

$$W_s(\theta) = \log \theta^\theta (1 - \theta)^{1-\theta} + \\ + \beta \min_{v_s} \left\{ (\phi_{s1} - \theta v_s) \left[ \frac{\log \lambda_1}{1-\beta} + W_1 \left( \theta \frac{\phi_{s1} - v_s}{\phi_{s1} - \theta v_s} \right) \right] + (\phi_{s2} + \theta v_s) \left[ \frac{\log \lambda_2}{1-\beta} + W_2 \left( \theta \frac{\phi_{s2} + v_s}{\phi_{s2} + \theta v_s} \right) \right] \right\}$$

for  $s = 1, 2$ . This is a two-dimensional dynamic programming problem that delivers optimal  $v_s$ ,  $s = 1, 2$ , as a function of  $\theta$ , and hence a law of motion for  $\theta$ .

### The decentralized economy

The problem of the consumer is

$$V_s(d, w, \theta) = \max_{c, b, e} \left\{ \log c + \beta \min_{v_s} \sum_{s'=1}^2 \phi_{ss'}(v_s) V_{s'}(\lambda_{s'} d, w'_{s'}, \theta'_{s'}) \right\}$$

subject to the budget constraint

$$c + p_s(d, \theta)e + q_s(d, \theta)b = w, \quad (77)$$

$$w'_{s'} = b + e [\lambda_{s'} d + p_{s'}(d\lambda_{s'}, \theta'_{s'})], \quad (78)$$

and the law of motion for  $\theta'_{s'}$  given by

$$\theta'_{s'} = g_{s'}(d, \theta, s) \quad (79)$$

where  $p$  is the price of equity,  $e$  is the fraction of the equity share held by the consumer,  $q$  is the price today of a bond that pays one unit of the consumption good next period, and  $b$  is the holdings of the bond. (The argument  $d$  is included for  $g$  only for completeness; it will not be there under the log assumption.)

The consumers' decision rules for all  $(d, w, \theta, s)$  are

$$c_{is}(d, w, \theta) \quad (80)$$

$$b_{is}(d, w, \theta) \quad (81)$$

$$e_{is}(d, w, \theta) \quad (82)$$

for  $i \in \{1, 2\}$ .

Total wealth in the economy when the state variable is  $(d, \theta, s)$  is  $d + p_s(d, \theta)$ . Thus, market clearing requires, for all values of the arguments,

$$c_{1s}(d, \theta [d + p_s(d, \theta)], \theta) + c_{2s}(d, (1 - \theta) [d + p_s(d, \theta)], \theta) = d \quad (83)$$

$$b_{1s}(d, \theta [d + p_s(d, \theta)], \theta) + b_{2s}(d, (1 - \theta) [d + p_s(d, \theta)], \theta) = 0 \quad (84)$$

$$e_{1s}(d, \theta [d + p_s(d, \theta)], \theta) + e_{2s}(d, (1 - \theta) [d + p_s(d, \theta)], \theta) = 1, \quad (85)$$

The relative wealth dynamics, finally, are given by

$$g_{s'}(d, \theta, s) = \frac{w'_{1s'}(d, \theta, s)}{w'_{1s'}(d, \theta, s) + w'_{2s'}(d, \theta, s)}, \quad (86)$$

where

$$w'_{1s'}(d, \theta, s) \equiv b_{1s}(d, \theta [d + p_s(d, \theta)], \theta) + e_{1s}(d, \theta [d + p_s(d, \theta)], \theta) (d\lambda_{s'} + p_{s'} [d\lambda_{s'}, g_{s'}(d, \theta)])$$

and

$$w'_{2s'}(d, \theta, s) \equiv b_{2s}(d, (1 - \theta) [d + p_s(d, \theta)], \theta) + e_{2s}(d, (1 - \theta) [d + p_s(d, \theta)], \theta) (d\lambda_{s'} + p_{s'} [d\lambda_{s'}, g_{s'}(d, \theta)])$$

Now, we will show how to find prices and portfolio allocations in this economy. We use the planning problem and identify the  $\theta$  in that problem with the corresponding variable here: the planning weight on agent 1 equals the relative fraction of total wealth held in equilibrium by this agent.

The prices of bonds,  $q_s(d, \theta)$ , and of equity,  $p_s(d, \theta)$ , then become

$$q_s(d, \theta) = \beta \left[ \phi_{s1}(v) \frac{\phi_{s1} - \theta v}{(\phi_{s1} - v)\lambda_1} + \phi_{s2}(v) \frac{\phi_{s2} + \theta v}{(\phi_{s2} + v)\lambda_2} \right] \equiv \hat{q}_s(\theta), \quad (87)$$

and

$$p_s(d, \theta) = \beta \left\{ \frac{(\phi_{s1} - \theta v) [\lambda_1 d + p_1(\lambda_1 d, \theta'_1)]}{\lambda_1} + \frac{(\phi_{s2} + \theta v) [\lambda_2 d + p_2(\lambda_2 d, \theta'_2)]}{\lambda_2} \right\} \quad (88)$$



We see that  $p_s(d, \theta) = d\hat{p}_s(\theta)$  solves this equation, delivering

$$\hat{p}_s(\theta) = \beta \{(\phi_{s1} - \theta v) [1 + \hat{p}_1(\theta'_1)] + (\phi_{s2} + \theta v) [1 + \hat{p}_2(\theta'_2)]\} \quad (89)$$

This is a system of two functional equations.

Asset holdings are the following. First, his equilibrium holdings of bonds are

$$b_{1s}(d, \theta) = [d\lambda_1 + p_1(\lambda_1 d, \theta'_1)] [\theta'_1 - e_{1s}(d, \theta)] = d\lambda_1 [1 + \hat{p}_1(\theta'_1)] [\theta'_1 - \hat{e}_{1s}(\theta)]$$

It is interesting to note here that bond holdings are proportional to  $d$ . Naturally, they are zero in the special case  $v = 0$ , when  $e = \theta$  and  $\theta' = \theta$ .

And his equilibrium holdings of equity are

$$\begin{aligned} e_{1s}(d, \theta) &= \frac{\theta'_1 [d\lambda_1 + p_1(\lambda_1 d, \theta'_1)] - \theta'_2 [d\lambda_2 + p_2(\lambda_2 d, \theta'_2)]}{[d\lambda_1 + p_1(\lambda_1 d, \theta'_1)] - [d\lambda_2 + p_2(\lambda_2 d, \theta'_2)]} = \\ &= \frac{\theta'_1 \lambda_1 [1 + \hat{p}_1(\theta'_1)] - \theta'_2 \lambda_2 [1 + \hat{p}_2(\theta'_2)]}{\lambda_1 [1 + \hat{p}_1(\theta'_1)] - \lambda_2 [1 + \hat{p}_2(\theta'_2)]} \equiv \hat{e}_{1s}(\theta) \end{aligned} \quad (90)$$

This is once more a system of two functional equations.

Neither bond holdings nor equity holdings depend directly on  $s$ , but they do through the dependence of the  $\theta$ 's on  $s$ .

### The special case where $\theta = 0$

We solve the problem for an ambiguity-averse agent who is measure zero in the economy. This agent solves the problem

$$V(w, d) = \max_{c, b, e} u(c) + \min_v \beta [(\phi - v)V(w'_1, d\lambda_1) + (1 - \phi + v)V(w'_2, d\lambda_2)]$$

subject to

$$c + qb + pde = w \quad (91)$$

$$w'_1 = b + (\lambda_1 d + p\lambda_1 d)e \quad (92)$$

$$w'_2 = b + (\lambda_2 d + p\lambda_2 d)e \quad (93)$$

The FOCs with respect to  $b$  are

$$\begin{aligned} &qu'(w - qb + pde) = \\ &= \beta \{(\phi - v)u' [b + e\lambda_1 d(1 + p) - qb' - pd\lambda_1 e'] + (1 - \phi + v)u' [b + e\lambda_2 d(1 + p) - qb' - pd\lambda_2 e']\} \end{aligned}$$

and with respect to  $e$ , they are

$$\begin{aligned} & pdu'(w - qb + pde) = \\ &= \beta \{(\phi - v)u' [b + e\lambda_1 d(1 + p) - qb' - pd\lambda_1 e'] \lambda_1 d(1 + p) + \\ &+ (1 - \phi + v)u' [b + e\lambda_2 d(1 + p) - qb' - pd\lambda_2 e'] \lambda_2 d(1 + p)\} \end{aligned}$$

Using logarithmic utility, we see that these equations become

$$q = \beta \left[ \frac{\phi - v}{b + e\lambda_1 d(1+p) - qb' - pd\lambda_1 e'} + \frac{1 - \phi + v}{b + e\lambda_2 d(1+p) - qb' - pd\lambda_2 e'} \right] (w - qb + pde)$$

$$\frac{p}{1+p} = \beta \left[ \frac{(\phi - v)\lambda_1}{b + e\lambda_1 d(1+p) - qb' - pd\lambda_1 e'} + \frac{(1 - \phi + v)\lambda_2}{b + e\lambda_2 d(1+p) - qb' - pd\lambda_2 e'} \right] (w - qb + pde).$$

We guess that

$$b = \alpha_b w \quad (94)$$

and

$$ed = \alpha_e w. \quad (95)$$

Then

$$q =$$

$$= \beta \left\{ \frac{\phi - v}{[\alpha_b + \alpha_e \lambda_1 (1+p)] (1 - q\alpha_b - p\lambda_1 \alpha_e)} + \frac{1 - \phi + v}{[\alpha_b + \alpha_e \lambda_2 (1+p)] (1 - q\alpha_b - p\lambda_2 \alpha_e)} \right\} (1 - q\alpha_b - p\alpha_e)$$

$$\frac{p}{1+p} =$$

$$= \beta \left\{ \frac{(\phi - v)\lambda_1}{[\alpha_b + \alpha_e \lambda_1 (1+p)] (1 - q\alpha_b - p\lambda_1 \alpha_e)} + \frac{(1 - \phi + v)\lambda_2}{[\alpha_b + \alpha_e \lambda_2 (1+p)] (1 - q\alpha_b - p\lambda_2 \alpha_e)} \right\} (1 - q\alpha_b - p\alpha_e)$$

The problem of the consumer can be rewritten as

$$V(w) = \max_{c, b, e} \log c + \min_v \beta [(\phi - v)V(w'_1) + (1 - \phi + v)V(w'_2)]$$

subject to

$$c + qb + p\hat{e} = w \quad (96)$$

$$w'_1 = b + \hat{e}\lambda_1(1+p) \quad (97)$$

$$w'_2 = b + \hat{e}\lambda_2(1+p), \quad (98)$$

where  $\hat{e} \equiv de$ . The variable  $v$  will be chosen (due to the envelope theorem) so that  $V(w'_1) = V(w'_2)$ , i.e., so that  $w'_1 = w'_2$ . That means that  $b = \alpha_b w$  and  $\hat{e} = 0$  – the agent does not hold equity – and from the FOC above, that

$$q = \beta \left[ \frac{\phi - v}{\alpha_b(1 - q\alpha_b)} + \frac{1 - \phi + v}{\alpha_b(1 - q\alpha_b)} \right] (1 - q\alpha_b). \quad (99)$$

This expression simplifies to

$$\alpha_b q = \beta \quad (100)$$

and then consumption is given by

$$c = (1 - \beta)w. \quad (101)$$

Since  $\frac{p}{1+p} = \beta$  in the  $\theta = 1$  case, this implies

$$\alpha_b = (\phi - v)\lambda_1 + (1 - \phi + v)\lambda_2. \quad (102)$$

Therefore,

$$v = \frac{\phi\lambda_1 + (1-\phi)\lambda_2 - \alpha_b}{\lambda_1 - \lambda_2} = \phi - \frac{\alpha_b - \lambda_2}{\lambda_1 - \lambda_2} \quad (103)$$

and

$$v = \phi - \frac{\alpha_b - \lambda_2}{\lambda_1 - \lambda_2}. \quad (104)$$

For  $\phi = 0.5$ ,  $\beta = 0.98$ ,  $\lambda_1 = 1.02$ , and  $\lambda_2 = 1.01$ ,  $\alpha_b = \frac{2\lambda_1\lambda_2}{\lambda_1 + \lambda_2}$ , and

$$v = \phi - \frac{\lambda_2}{\lambda_1 + \lambda_2} = 0.00246. \quad (105)$$

## A2 Proofs of subsection 5.3

We want to proof that

$$E(\theta' | \theta) < \theta. \quad (106)$$

Since

$$E(\theta' | \theta) = \phi \frac{\theta(\phi - v)}{\phi - \theta v} + (1 - \phi) \frac{\theta(1 - \phi + v)}{1 - \phi + \theta v}, \quad (107)$$

expression (106) becomes:

$$\phi \frac{\theta(\phi - v)}{\phi - \theta v} + (1 - \phi) \frac{\theta(1 - \phi + v)}{1 - \phi + \theta v} < \theta. \quad (108)$$

Simplifying (108) yields:

$$\theta^2 v^2 < \theta v^2. \quad (109)$$

Since  $v \neq 0$ ,

$$\theta < 1. \quad (110)$$

And condition (110) is always true in the case we study, otherwise we would be back to the case of one agent.

The proof that  $\lim_{\theta \rightarrow 0} E\left(\frac{\theta'}{\theta}\right) = 1$  is even simpler. First, consider expression for  $E\left(\frac{\theta'}{\theta}\right)$ :

$$E\left(\frac{\theta'}{\theta}\right) = \phi \frac{\phi - v}{\phi - \theta v} + (1 - \phi) \frac{1 - \phi + v}{1 - \phi + \theta v}. \quad (111)$$

Then, the limit becomes:

$$\begin{aligned} \lim_{\theta \rightarrow 0} \phi \frac{\phi - v}{\phi - \theta v} + (1 - \phi) \frac{1 - \phi + v}{1 - \phi + \theta v} = \\ \phi \frac{\phi - v}{\phi} + (1 - \phi) \frac{1 - \phi + v}{1 - \phi} = \phi - v + 1 - \phi + v = 1. \end{aligned} \quad (112)$$

## 6.1 A3 Figures

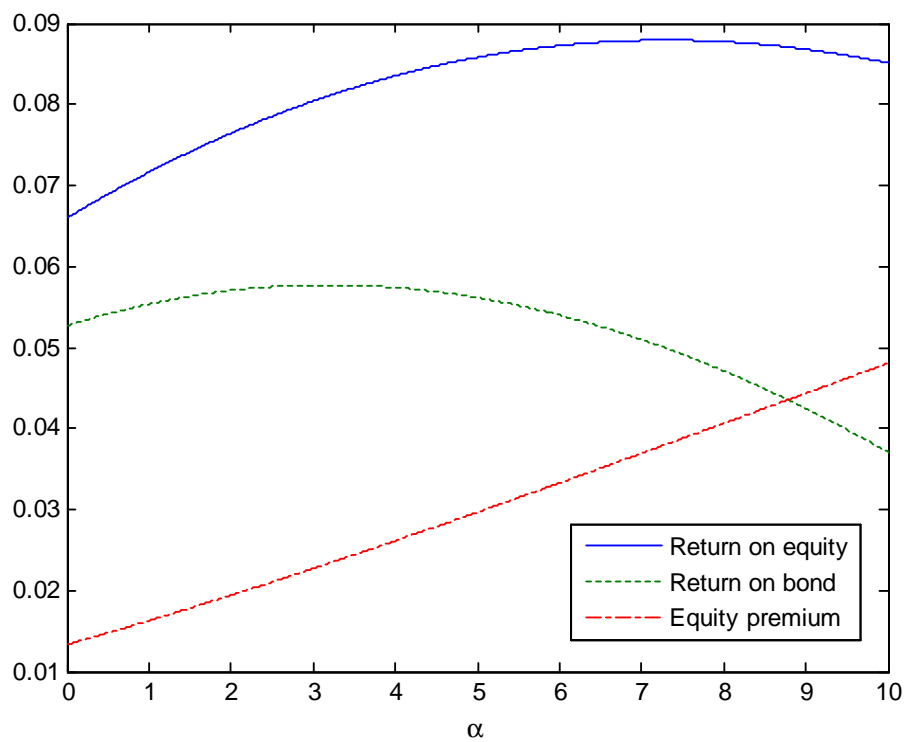


Figure 1: Return on equity, risk-free return, and the equity premium as a function of the risk aversion parameter ( $\alpha$ )

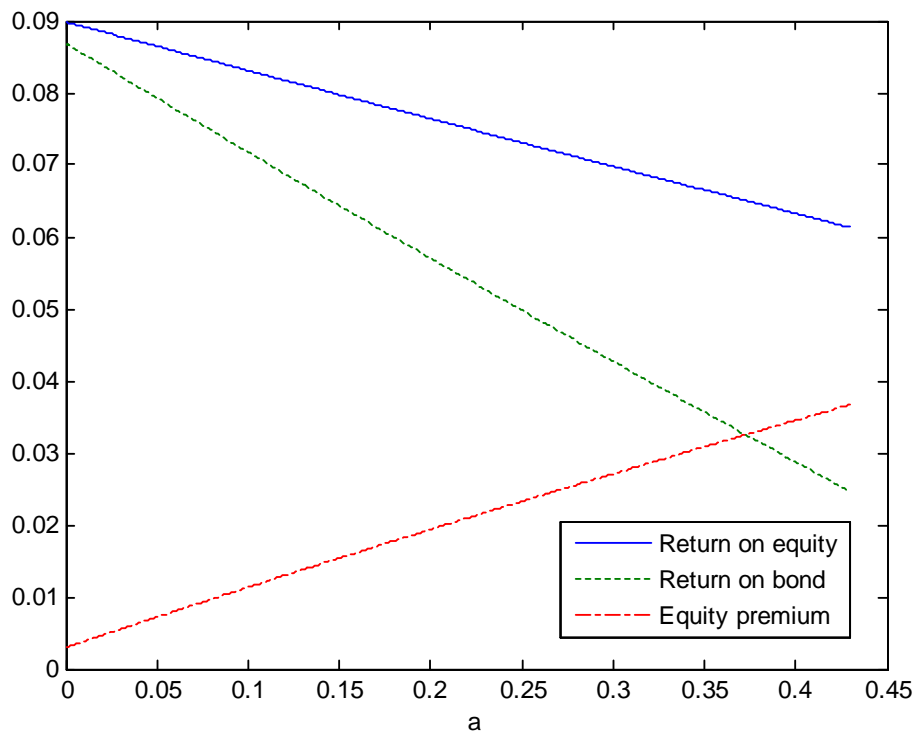


Figure 2: Return on equity, risk-free return, and the equity premium as a function of the ambiguity aversion parameter ( $a$ )

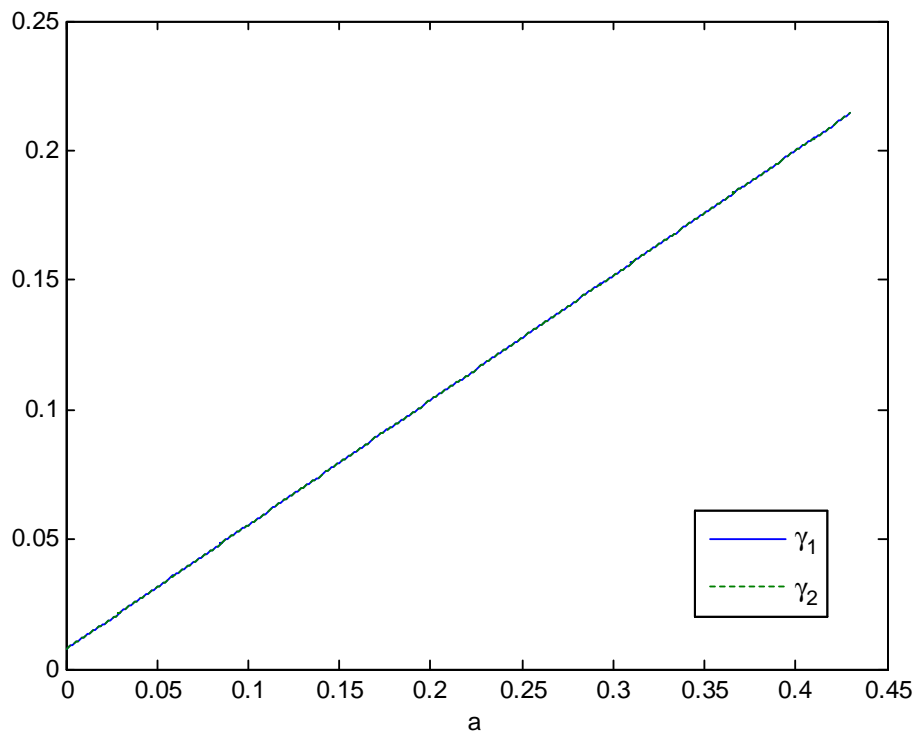


Figure 3: Costs of business cycles as a function of the ambiguity-aversion parameter ( $a$ )

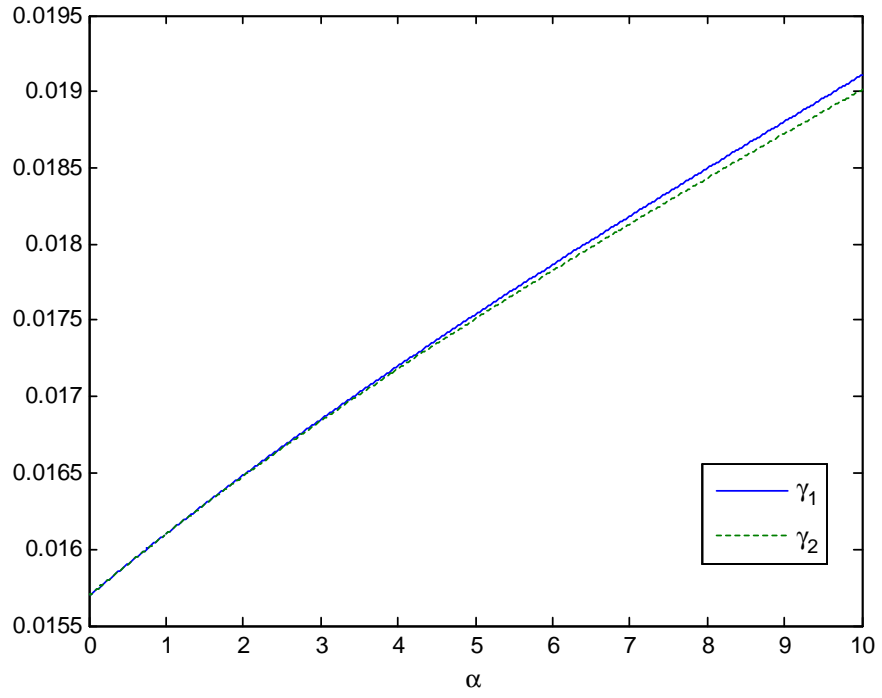


Figure 4: Cost of business cycles as a function of the risk-aversion parameter ( $\alpha$ )

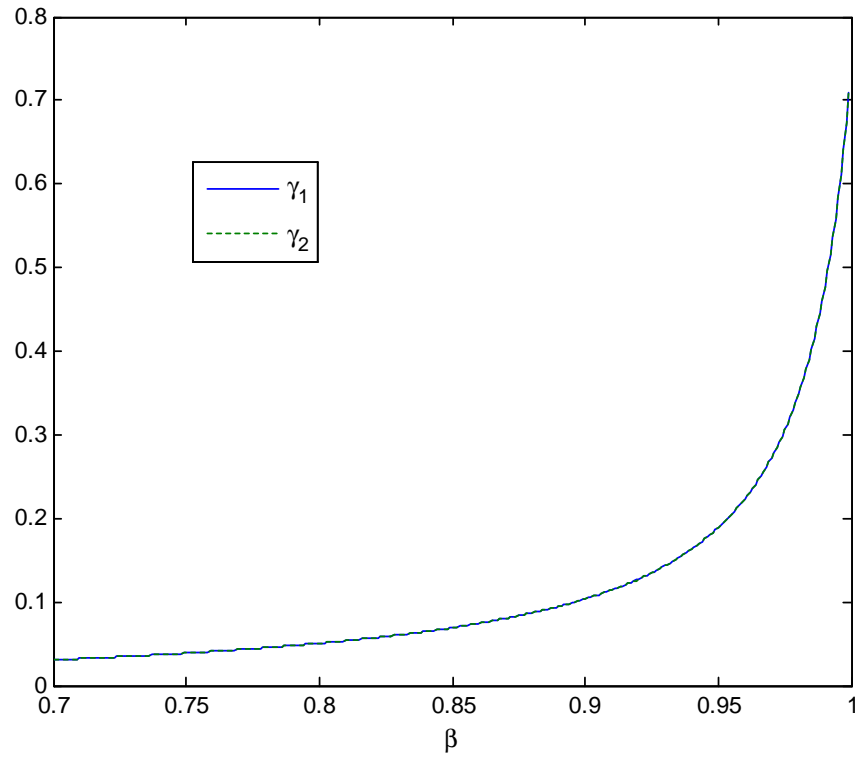


Figure 5: Cost of business cycles as a function of the subjective discount factor ( $\beta$ )



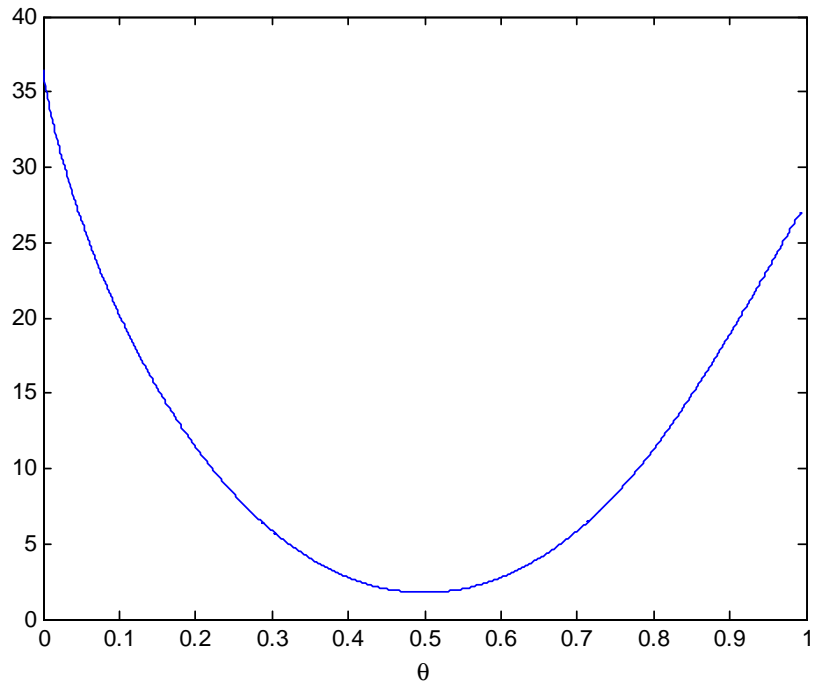


Figure 6: Value function  $W(\theta)$

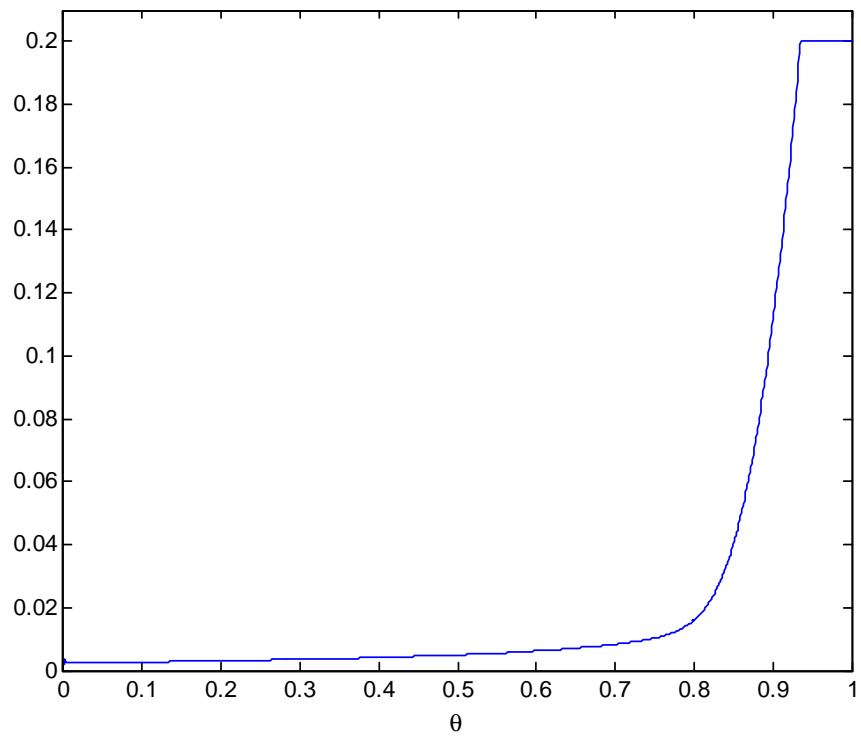


Figure 7: Policy function  $v(\theta)$

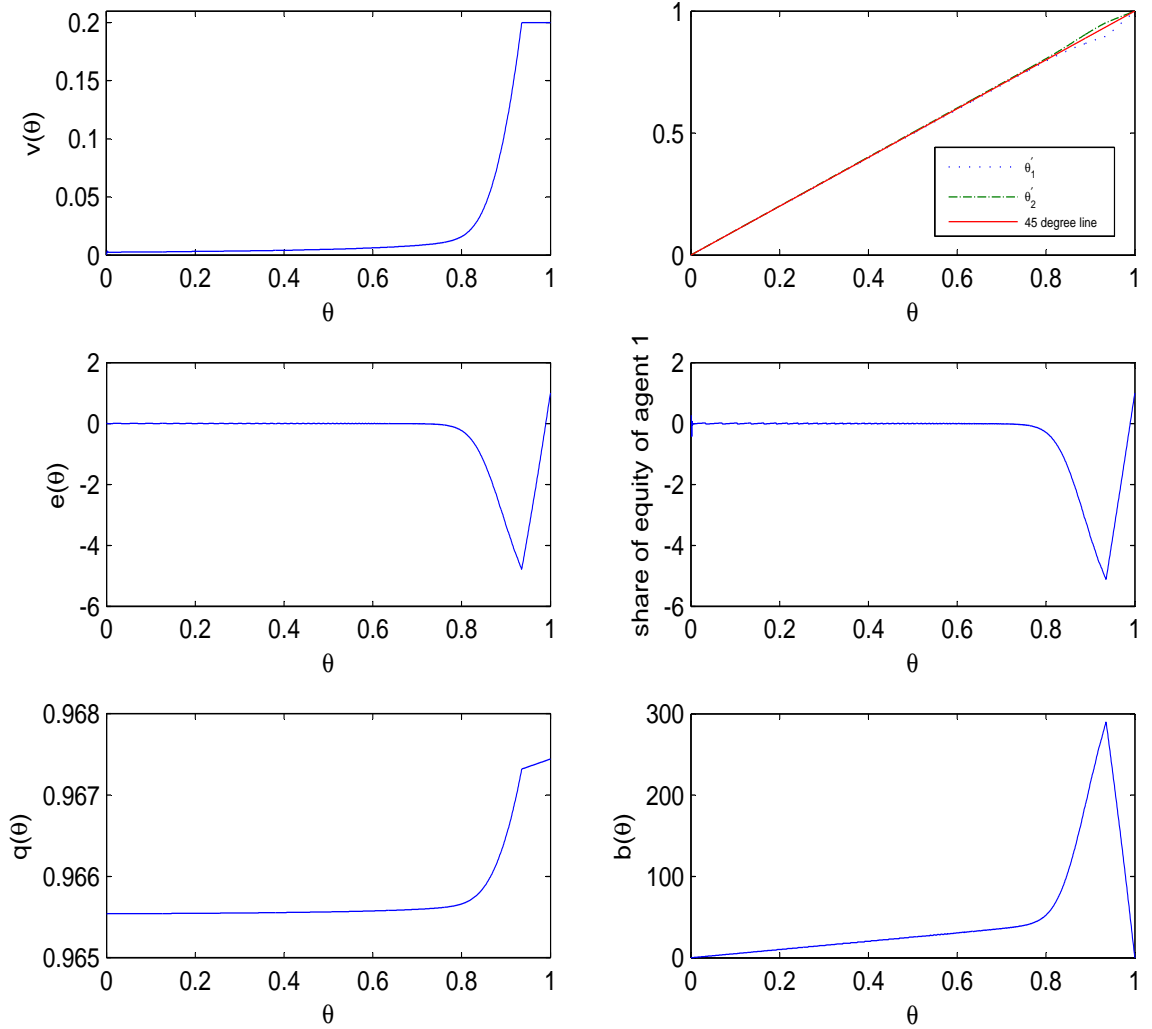


Figure 8: From left to right from top to bottom: (a)  $v(\theta)$ , (b) law of motion for  $\theta$  ( $\theta_1$  below the 45 degree line;  $\theta_2$  above the 45 degree line), (c)  $e_1(\theta)$ , (d) share of equity of agent 1, (e)  $q(\theta)$ , (f)  $b(\theta)$

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