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THE VON NEUMANN TURNPIKE UNDER UNCERTAINTY:

2-Sector Case

by

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I. Introduction

There now exist substantial literatures on von Neumann economies, especially their turnpike properties [7, 9, 10], and on optimal growth theory under uncertainty [1, 2, 8]. A typical deterministic von Neumann model has a rather complicated production technology, while probabilistic optimal growth models usually have fairly simple production technologies. The present work attempts to join these two literatures within the confines of a two-sector von Neumann model under uncertainty. The primary interpretation of the model is that of an optimally planned economy [3, 4].

In the 1-sector von Neumann model, it is easy to show that the introduction of a multiplicative random disturbance with mean one has no affect on the allocation of resources for final value maximization. Further, an increase in risk (in the sense of [13]) associated with such a random disturbance does not affect the expected growth rate of the system, which in all cases equals the deterministic maximum. These results explain the need for going to a 2-sector model. Here, the paper shows two results, quite different from those in the 1-sector model. First, only if the random disturbance is small, is the modified deterministic decision rule optimal under uncertainty. (For a detailed statement, see Theorem 3.) Second, the expected steady-state rate of growth is always less, under uncertainty, than the deterministic expansion rate, the expected growth loss being greater the greater the risk. These results imply that increases in risk will have a noticeable impact on the pattern of resource allocation and on the observed rate of growth of an optimally planned von Neumann economy.

The paper is organized as follows. Section 2 disposes of the one-sector model. The deterministic two-sector model, and especially its turnpike, is the subject of section 3. A dynamic programming approach to this model under
uncertainty is undertaken in section 4. Section 5 investigates the steady-state of the dynamic programming model in the small disturbance case. Some suggestions for further research conclude the paper.

II. The One-Sector Model

This section reviews a paradigm case of the one-sector model, namely linear production with unit period of production. Thus, if $F^t$ is free resources in period $t$, and $P^t$ is projects started in period $t$, then

$$F^{t+1} = F^t - a P^t + \alpha_t \beta^t,$$

where $a < 1$, is the input-output requirement. In the deterministic case, $\alpha_t = 1$. The planning objective is terminal value maximization

$$\max F^{T+1}$$

subject to the transaction equation (1), the initial condition (3),

$$f^0$$
given,

and the constraint

$$0 \leq P^t \leq \frac{F^t}{a}$$

It is clear by a standard dynamic programming argument that the optimal policy satisfies

$$P^t = \frac{F^t}{a},$$

so that the economy expands at the rate of $\frac{F^{t+1}}{F^t} = \frac{1}{a} = \lambda$, the von Neumann expansion rate. The value of the optimal plan, from (2), (3), and (5) then is

$$\max F^{T+1} = (\lambda)^{T+1} F^0.$$
Results are not greatly different in the uncertainty case, where $\alpha_t$ is an independently identically distributed random variable with $E\alpha_t = 1$. The planning objective is now expected terminal value maximization

\[ (2') \quad \max E F^{T+1} \]

subject to the same conditions as before. The optimal policy still satisfies (5), and the expected expansion rate $\frac{E(F^{t+1})}{F^t} = \lambda$ as before. Finally, the value of the optimal plan, assuming independence of the various realizations of $\alpha$, is

\[ (6') \quad \max E(F^{T+1}) = (\lambda)^{T+1} e^{\sum_{i=0}^{T} \alpha_i} = (\lambda)^{T+1} e^{0} \]

Thus the uncertainty ultimately has no apparent impact on the planning problem. It might be expected that the results of the one-sector model would generalize to the two-sector model. Such is the case for the deterministic model, but rather drastic differences thwart the generalization in the event of uncertainty.

III. The Two-Sector Model under Certainty

The model of the previous section is extended to two-sectors by the addition of an intermediate goods sector. Each project requires two periods for completion, with $a$ units of resources required as input in each period. Productiveness of the system requires $a < 1/2$, which will henceforth be assumed. Thus, one now distinguishes between $\hat{F}^t$, projects updated and $N^t$, new projects started, in period $t$. The transition equations become

\[ (7) \quad F^{t+1} = F^t - a\hat{F}^t - aN^t + \alpha_t \hat{F}^t \]

\[ (8) \quad P^{t+1} = P^t - \hat{P}^t + N^t \]
where \( P^t \) represents the intermediate good. The choice between starting new projects and updating old ones is reflected in the constraints

\[
(9) \quad 0 \leq \dot{P}^t \leq \min \left( \frac{F^t}{a}, P^t \right)
\]

since not more projects can be updated than are in process, and

\[
(10) \quad 0 \leq N^t \leq \frac{F^t}{a} - \dot{P}^t
\]

The objective is to

\[
(11) \quad \max F^{T+1} + aP^{T+1}
\]

work in progress being evaluated at its resource cost.

In the deterministic case when \( \alpha_t = 1 \), the planning problem then is to maximize (11), subject to (7)-(10) and the initial conditions

\[
(12) \quad F^0, P^0 \text{ given}
\]

Since the objective function exhibits constant returns to scale, it is useful to define new variables that reflect intensities:

\[
(13) \quad x^t = \frac{F^t}{aP^t}
\]

\[
(14) \quad \theta^t = \frac{\dot{P}^t}{P^t}
\]

\[
(15) \quad \gamma^t = \frac{N^t}{P^t}
\]

Rewriting (7) through (10) in the form,

\[
(7') \quad x^{t+1} = \frac{\alpha_t \theta^t + a(x^t - \theta^t - \gamma^t)}{a(1 - \theta^t + \gamma^t)}
\]

\[
(8') \quad P^{t+1} = P^t(1 - \theta^t + \gamma^t)
\]
\((9')\) \(0 \leq \theta^t \leq \min (1, x^t)\)

\((10')\) \(0 \leq \gamma^t \leq x^t - \theta^t\)

We seek the maximum of (11) subject to \((7')-(10')\) and (12). Let us call this problem in the deterministic case \((\alpha_t = 1)\) problem I.

**Theorem 1.** There exists an optimal solution \((\theta^*_t, \gamma^*_t)\) to problem I. It is characterized by

(a) \(x^t \leq x_L\) then \(\theta^*_t = x^t, \gamma^*_t = 0\)

(b) \(x^t \in (x_L, x_U)\) then \(0 < \theta^*_t < \min (1, x^t)\)

\[\gamma^*_t > 0\]

Satisfying

\[\theta^*_t + \gamma^*_t = x^t\]

and \(x^{t+1} = x_U^t\)

(c) \(x^t \geq x_U^t\) then \(\theta^*_t = 1, \gamma^*_t = 0\)

and any \(\gamma^*_t\) such that \(x^{t+1} \in [x_L^{t+1}, x_U^{t+1}]\).

\(x_L^t, x_U^t\) are defined recursively as

\[(16)\]

\[x_U^t = \frac{1 + ax^{t+1}_U}{ax_U^{t+1}}\]

with the initial conditions \(x_T^T = x_L^T = 1\).

\[x^t_L = 1/x_U^t\]

Furthermore, as \(T\) grows large \(\lim x_U^t = x^*\), the von Neumann ray.
Proof:

First we show the optimality of the decision rule; define the stage return function:

\[(17) \quad J^t(F^t, P^t) = \max \left\{ J^{t+1}(F^{t+1}, P^{t+1}) \mid \hat{F}^t, \hat{N}^t \right\} \]

subject to (7)-(10). The optimality of the decision rule is established by proving, by induction, that the stage return function has the following properties:

(18a) \( J^t \) is a continuous, linear homogeneous function in \( F^t, P^t \); furthermore, it is concave and piecewise linear.

\[(18b) \quad J^t(F^t, P^t) \text{ has the following form: } \frac{1}{J^t} \]

\[i) \quad \left( \frac{J^t}{J_F} \right) = \left( \frac{1-a}{a} \right) \quad \text{for} \quad x^t < x_L^t; \quad J_Y^t \equiv \frac{\partial J^t}{\partial y} \]

\[ii) \quad J^t = P^t \cdot b_1^t (1 + x^t); \quad \text{for} \quad x^t \in [x_L^t, x_U^t] \]

\[iii) \quad J^t = P^t \cdot b_1^t \left( \frac{1-a}{a} + x^t \right) \quad \text{for} \quad x^t > x_U^t. \]

where, by concavity, \( b_2^t \left( \frac{1-a}{a} \right) > b_1^t > b_2^t. \)

Property (18a) is rather obvious, once we note that \( J^{T+1} = F^{T+1} + aP^{T+1} \) is linear, that all constraint and feedback equations (7-10) are linear, and that the feasible set (\( \hat{F}, \hat{N} \)) for any \( t \) is convex. It is important to note that \( J^t \) (for \( t < T+1 \)) is concave, but not convex since it is not strictly linear.

The proof of (18b) is by induction. Rather obviously, the optimal strategy at \( T \) is: \( F^T = \min \{ F^T/a, P^T \} \); \( N^T \) indeterminate (\( \varepsilon^T = \min \{ 1, x^T \} \), \( \gamma^T \) indeterminate). Thus, \( J^T \) is given by:
\[
J^T = \begin{cases} 
\alpha^T (1-a)x^T + a; & (J^T/F^T) = ((1-a)/a^2), \quad x^T < 1 \\
(P^T/2)(1 + x^T) & , \quad x^T = 1 \\
\alpha^T [(1-a) + ax^T] & , \quad x^T > 1.
\end{cases}
\]

which has the required form.

Assuming \(J^{t+1}\) has the required form, we show \(J^t\) also has this form by establishing the optimality, at \(t\), of the decision rule given by Theorem 1.

Since \(J^{t+1}\) is strictly increasing in \((P^{t+1}, F^{t+1})\) it is obvious that either:

(i) \(a(\hat{P}^t + N^t) = F^t\), or (ii) \(\hat{P}^t = P^t\). Next, note that if \(dN^t = -d\hat{P}^t\), then:

\[
dF^{t+1} = (d\hat{P}^t), \quad dP^{t+1} = -2(d\hat{P}^t);
\]

and for \(d\hat{P}^t = 0,\)

\[
dF^{t+1} = -adN^t, \quad dP^{t+1} = dN^t.
\]

From (7') and (16), note that if \(x^t < x_L^t\), then \(x^{t+1} < x_U^{t+1}\), regardless of \(\theta^t, \gamma^t\); from (18bii), \(J_{F^t}^{t+1} > 2J_{P^t}^{t+1}\) for \(x^{t+1} < x_U^{t+1}\); thus, from (20), we see \(\gamma^t = 0, \quad \theta^t = x^t\) for \(x^t < x_L^t\).

Next, assume \(x^t \in (x_L^t, x_U^t)\); we show that, for an optimal rule, \(x^{t+1} = x_U^{t+1}\).

Note that \((J_{F^t}^{t+1} - 2J_{P^t}^{t+1}) > 0\) as \(x^{t+1} = x_U^{t+1}\). Thus, \(x^{t+1} > x_U^{t+1}\) cannot be optimal since choosing \(dN^t = -d\hat{P}^t > 0\) is feasible, increases \(J^{t+1}\), and decreases \(x^{t+1}\). Similarly, \(x^{t+1} < x_U^{t+1}\) cannot be optimal since (i) if \(\hat{P}^t = \min[F^t, a, P^t]\), then \(x^{t+1} < x_U^{t+1}\), a contradiction; and (ii) for \(\hat{P}^t < \min[F^t, a, P^t]\), \(N^t > 0\), and \(J^{t+1}\), \(x^{t+1}\) are increased by choosing \(d\hat{P}^t = -dN^t > 0\). Thus, for \(x^t \in (x_L^t, x_U^t)\), \(\theta^t, \gamma^t\) are chosen such that \(x^{t+1} = x_U^{t+1}\).

Finally, if \(x^t > x_U^t\), then \(\theta^t < 1\) implies \(x^{t+1} < x_U^{t+1}\), which cannot be optimal, as argued above (let \(d\hat{P}^t = -dN^t > 0\); therefore \(\theta^t = 1\). For \(x^{t+1} < x_L^t\), then, from the form of \(J^{t+1}\), \(\frac{dJ^{t+1}}{dN^t} < 0\); for \(x^{t+1} > x_U^t\), \(\frac{dJ^{t+1}}{dN^t} > 0\), whereas...
for \( x_{t+1} \in (x_L^{t+1}, x_U^{t+1}) \), \( \frac{dN^t}{dt} = 0 \). Thus, \( N^t(\gamma^t) \) is indeterminate, but must be such that \( x_{t+1} \in (x_L^{t+1}, x_U^{t+1}) \).

This shows Theorem 1 holds, if \( J^t \) has the required form. Finally, we show \( J^t \) must have this form if \( J^{t+1} \) does. For \( x^t < x_L^t, x^{t+1} < x_U^{t+1} \),

\[
\hat{p}^t = F^t/a, \quad N^t = 0; \text{ thus:}
\]

\[
\frac{\partial J^t}{\partial p^t} = \frac{\partial J^{t+1}}{\partial p^{t+1}} \cdot \frac{\partial F^{t+1}}{\partial p^t} + \frac{\partial J^{t+1}}{\partial p^{t+1}} \cdot \frac{\partial p^{t+1}}{\partial p^t} = \frac{\partial J^{t+1}}{\partial p^{t+1}};
\]

\[
\frac{\partial J^t}{\partial F^t} = \frac{\partial J^{t+1}}{\partial F^{t+1}} \cdot \frac{\partial F^{t+1}}{\partial F^t} + \frac{\partial J^{t+1}}{\partial F^{t+1}} \cdot \frac{\partial F^{t+1}}{\partial p^{t+1}} = \left( \frac{1}{a} \right)^2 \left[ \frac{\partial J^{t+1}}{\partial F^{t+1}} - \frac{\partial J^{t+1}}{\partial F^{t+1}} \right]; \text{ and}
\]

\[
\frac{J^t}{\hat{p}^t} = \frac{J^{t+1}}{\hat{p}^{t+1}} = \left( \frac{1}{a} \right)^2 + \frac{1}{a^2} - \frac{1}{a} = \left( \frac{1-a}{a^2} \right), \text{ since } \frac{J^t}{\hat{p}^t} \geq \frac{1}{a}, \quad x^{t+1} < x_U^{t+1}.
\]

For \( x^t \in (x_L^t, x_U^t) \), \( x^{t+1} = x_L^{t+1} \), and we have, using (7'), (8'), and (18b):

\[
J^t = p^t b_1^{t+1} \left( \frac{1 + x_U^{t+1}}{1 + 2a x_U^{t+1}} \right) (1 + x^t)
\]

For \( x^t > x_U^t \), \( \hat{p}^t = 1 \), \( x^{t+1} \in (x_L^{t+1}, x_U^{t+1}) \) and:

\[
J^t = p^t \cdot b_1^{t+1} \left( \frac{(1-a)}{a} + x^t \right)
\]

From (23)-(25), we see \( J^t \) has the appropriate form, completing the proof of the optimality of the decision rule.

To show \( x_U^t \to x^* \), where \( x^* \) is the von Neumann ray, first observe that \( x^* = \left( \frac{1}{a} + \sqrt{\frac{1}{4} + \frac{1}{a}} \right) \), where \((x^* - 2) > 0 \) (a < 1/2) is the growth rate of the system. From (16):
\[ x_U^t = \frac{1 + ax_U^{t+1}}{ax_U^{t+1}} \equiv \left( \frac{A}{B} \right)^t, \text{ where we define } x^t \equiv (A^t / B^t). \]

Hence,

\[ \frac{A}{B}^t = \left( \frac{B^{t+1} + aA^{t+1}}{aA^{t+1}} \right); \quad B^t = aA^{t+1}, \quad A^t = B^{t+1} + aA^{t+1}, \text{ or} \]

\[ (26') \quad A^t = a[A^{t+1} + A^{t+2}] \]

Thus, (26') is a second order difference equation, with initial conditions
\[ A^T = 1, \quad (B^T = 1) \quad A^{T-1} = (1 + a). \] Solving (26') yields:

\[ A^t = c_1 \lambda_1^t + c_2 \lambda_2^t, \quad t \leq T; \text{ where } \lambda_1 = -\left[ \frac{1}{2} \pm \sqrt{\left( \frac{1}{4} + \frac{a}{4} \right)} \right]; \text{ or:} \]

\[ \lambda_1 = -(x^*), \quad \lambda_2 = (x^* - 1), \quad c_1 = -\frac{(x^* - 1)^2}{(2x^* - 1)(-x^*)}; \quad c_2 = \frac{(x^*)^2}{(2x^* - 1)(x^* - 1)} \]

Thus, as \( t \to -\infty \),

\[ x_U^t = \left( \frac{A}{aA^{t+1}} \right) \to \left( \frac{1}{a\lambda_2} \right) = x^*. \]

This completes the proof of Theorem 1.

Intuitively, the optimal policy is quite simple: at time \( t \), aim the economy at \( x_U^{t+1} \). The further the termination date \( T \) is, the closer is \( x_U^{t+1} \) to the von Neumann ray.\(^2\) Thus, near the beginning of a very long plan, the model has the familiar turnpike property. Indeed, the horizon need not be too long for the target to be near the turnpike: for growth rates of 20% or less, \( |x_U^t - x^*| < .001 \) for all \( t \) such that \( T - t \geq 12. \)
IV. The Two-Sector Model under Uncertainty: Solution at T-1

This section considers the impact of uncertainty on the two sector model of the last section. The only formal differences are that $\alpha_t$, instead of being a constant equal to one in equation (7'), is now a random variable satisfying

$$(29) \quad \alpha_t = \begin{cases} 
1 - \varepsilon & \text{with probability } z \\
1 & \text{with probability } 1 - 2z \\
1 + \varepsilon & \text{with probability } z 
\end{cases}$$

$$0 \leq \varepsilon \leq 1, \ 0 \leq z \leq 1/2$$

$$E[(\alpha_t - 1)(\alpha_{t-1} - 1)] = 0 \quad \text{for } i \neq 0$$

and equation (11) is replaced by its expectation. The parameter $\varepsilon$ measures the size of the disturbance; the parameter $z$ measures the probability of a disturbance; the cases $\varepsilon = 0$ or $z = 0$ correspond to certainty. An increase in $z$ is a mean-preserving spread of $\alpha$, hence an increase in risk. To find the optimal solution, introduce the Lagrangean $L^t$:

$$(30) \quad L^t = E[J^{t+1}_F] + \lambda_1 (F^t - \hat{F}^t) + \lambda_2 (F^t - a_F^t - aN^t).$$

From (7), (8), and (17), the first order conditions for an optimum are given by

$$(31) \quad L^t_p = E[J^{t+1}_F (\alpha - a) - J^{t+1}_p] - \lambda_1 - \lambda_2 a \leq 0$$

$$(32) \quad L^t_N = E[J^{t+1}_p - aJ^{t+1}_F] - \lambda_2 a \leq 0 ;$$

where $J^{t+1}_F = \frac{\partial J^{t+1}_F}{\partial t^{t+1}_F}$, $J^{t+1}_p = \frac{\partial J^{t+1}_p}{\partial t^{t+1}_p}$.

As before, let $J^{T+1} = F^{T+1} + aP^{T+1}$; hence, $J^{T+1}_F$ and $J^{T+1}_p$ are constant, and
the optimal policy at time $T$ is:

\[(33) \quad \hat{p}_T^* = \min(p_T^*, F_T^*/a)\]

Since this is the same policy as for the certainty solution, the optimal return function is again given by (19).

While the uncertainty has no impact on the optimal decision or return function at $T$, this need not be true at $T-1$, since $J_T^*$ is concave and piecewise linear. Suppose the planner still aims at $x_U^T$. There now exists the choice between aiming based on the average outcome (denoted $x_U^T = x^T(1)$), aiming based on the favorable outcome ($x_U^T(1 + \varepsilon)$), and aiming based on the unfavorable outcome ($x_U^T(1 - \varepsilon)$). Indeed, the policy that aims at the target on average is just the deterministic policy in the probabilistic setting.

One can now state and prove the following result:

**Theorem 2.** The optimal policy rule at time $(T-1)$ is the same as that in Theorem 1 if and only if $z < z^*$, where $z^* = a/(1 + 2a - \varepsilon)$ is less than 1/2. Furthermore, the optimal return under uncertainty is less than or equal that under certainty, with strict inequality when $a/(1 + a + \varepsilon) < x_U^{T-1}$.

**Proof:**

Since $J_T^*$ is the same as for Theorem 1, the proof follows that used earlier, the difference being that the realized value of $x_T^*$ depends on $\alpha$. From (19), (29) we note:

\[(34) \quad E[\alpha J_F^T - 2J_F^T] > 0, \quad x_T^*(1) < 1;\]

\[E[\alpha J_F^T - 2J_F^T] < 0 \text{ for } x_T^*(1 - \varepsilon) < 1 < x_T^*(1) \text{ or } z < z^*.\]

Assume $z < z^*$; then \(\text{sign } \{E[\alpha J_F^T - 2J_F^T]\} = \text{sign } \{1 - x_T^*(1)\};\) i.e., attention is
focused on the "average" outcome, $x_T^T(1)$. By earlier reasoning, if $x_T^T(1) < 1$, $p_T^{-1} < p_T^{-1}$, then $E[J_T^T]$ can be increased by increasing $p_T^{-1}$, decreasing $N_T^{-1}$; for $x_T^T(1) > 1$, $E[J_T^T]$ can be increased by decreasing $p_T^{-1}$ and increasing $N_T^{-1}$.

Thus, for $x_T^T(1) < x_U^T(1) = (1 + a) / a$, $p_T^{-1} < p_T^{-1}$, and the decision rule is identical to that for the deterministic case. Similarly, $x_T^T(1) > x_U^T(1)$, $p_T^{-1} = p_T^{-1}$, then, regardless of $z$, $E[J_T^T - aJ_F^T] > 0$ as $x_T^T(1) < 1$. Thus, for $z < z^*$, the optimal decision rule at $(T-1)$ is unaltered by the uncertainty.

For $z > z^*$, by the prior argument, the decision rule is unaffected for $x_T^T < x_L^T$ (so that $x_T^T(1) < x_U^T$) and for $x_T^T > x_L^T$ (so that $x_T^T(1) = 1$).

However, since $z > z^*$ implies $E[\alpha J_T^T - 2J_F^T] > 0$ for $x_T^T(1 - \varepsilon) < 1 < x_T^T(1)$, this implies $E[J_T^T]$ can be increased by increasing $p_T^{-1}$, decreasing $N_T^{-1}$.

Thus, the outcome $\alpha = (1 - \varepsilon)$ is used and is steered towards the target

$x_U^T = 1$. Since $E[\alpha J_F^T - 2J_F^T] < 0$ for $x_T^T(1 - \varepsilon) > 1$, it is clear $N_T^{-1} > 0$ for $x_T^T > \frac{a}{1 + a - \varepsilon} > \frac{a}{1 + a}$. For $x_T^T \in \left[\frac{a}{1 + a - \varepsilon}, \frac{1 + a - \varepsilon}{a}\right]$, $p_T^{-1}$, $N_T^{-1}$ are chosen such that $x_T^T(1 - \varepsilon) = x_U^T = 1$. For $x_T^T \in \left(\frac{a}{1 + a - \varepsilon}, \frac{1 + a}{a}\right)$, $p_T^{-1} = p_T^{-1}$ and $N_T^{-1} = (F_T^{-1}/a - p_T^{-1})$, since $x_T^T(1) > 1$. Thus, the uncertainty modifies the optimal solution if $x_T^T \in \left(\frac{a}{1 + a}, \frac{1 + a}{a}\right)$, $z > z^*$; and it leads to more emphasis on completing existing projects, rather than starting new ones.

The function $J_T^{-1}$ is found by routine calculation, given the optimal decision rule. Note that since $J_T^T$ is not strictly linear, it is not surprising that the uncertainty reduces $E[J_T^T]$. These results are presented in Table 1.

From Table 1 it is readily seen $J_T^{-1}_{z_1}$, $J_T^{-1}_{\varepsilon} < 0$ unless $x_T^{-1} < \frac{a}{1 + a + \varepsilon}$ (i.e., $x_T^T(1 + \varepsilon) < x_U^T = 1$). This completes the proof of Theorem 2.

Several things are noteworthy from Theorem 2. First, if the uncertainty has any impact, it is to increase the emphasis on completing existing projects at the expense of new ones. It does not affect the choice between starting new projects or holding inventories. Second, a mean-preserving spread of the
\[
\begin{array}{|c|c|c|}
\hline
x_{T-1}^* & z<z^* & z>z^* \\
\hline
\frac{a}{1+a+\varepsilon} & J_{T-1} = p_{T-1}^* \left[ \left( \frac{1-a-\varepsilon}{a} \right)x_a + a \right] & \text{same as } z<z^* \\
\frac{a}{1+a+\varepsilon}, \frac{a}{1+a-\varepsilon} & J_{T-1} = p_{T-1}^* \left[ \left( \frac{1-a-\varepsilon}{a} \right)x_a + a \right] & \text{same as } z<z^* \\
\frac{1+a-\varepsilon}{a}, \frac{1+a}{a} & J_{T-1} = p_{T-1}^* \left[ \left( \frac{1-a+\varepsilon}{a} \right)x_a + a \right] & \text{same as } z<z^* \\
\frac{1+a}{a} & J_{T-1} = p_{T-1}^* \left[ \left( \frac{1-a+\varepsilon}{a} \right)x_a + a \right] & \text{same as } z<z^* \\
\hline
\end{array}
\]

Note: \( J_{z_{T-1}}^* \leq 0, J_{\varepsilon_{T-1}}^* \leq 0 \) everywhere; \( x \equiv x_{T-1}^* \).
distribution raises the likelihood that the uncertainty will affect the resource allocation. The larger the probability of the worst outcome, the more likely it is the optimal plan will aim the economy toward $x^T_U = x^T (1-\varepsilon)$. An increase in $\varepsilon$ has ambiguous effects. On the one hand, it increases $z^*$, and thus the a priori likelihood that the deterministic policy will be optimal. On the other hand, if the economy is being aimed at $x^T_U = x^T (1-\varepsilon)$, then more weight is given to completing projects to insure the target is met. Finally, an increase in $\alpha$ raises $z^*$. Since $\alpha$ is inversely related to the expansion rate of the system, the larger the deterministic expansion rate of the system, the more likely it is that uncertainty affects the resource allocation.

V. The Steady-State under Uncertainty

Although the complete solution of the model of the last section can be carried beyond $T-1$, the computations become quite burdensome. This difficulty can be circumvented however by a direct appeal to steady-state properties. For instance, we have already seen that the target $x^t_U$ approaches very quickly to the von Neumann ray $x^*$. Moreover, we have seen that, even starting from terminal value maximization, the value function $J^t$ is concave and piecewise linear. In light of the above, we postulate the value function $J^T$ as (compare (18b))

\[
J^T(P^T, P^T) = \begin{cases} 
  p^T b_0^T \left( \frac{1-a}{a} x^T + 1 \right), & x^T < (x^*)^{-1} \\
  p^T b_1^T (x^T + 1), & (x^*)^{-1} \leq x^T \leq x^* \\
  p^T b_2^T (x^T + \frac{1-a}{a}), & x^* < x^T
\end{cases}
\]

(35)

where, by continuity, $b_2^T = b_0^T = \left[ \frac{a(1+x^*)}{1-a+ax^*} \right] b_1^T < b_1^T$. 
One notes that, compared to (18b), (35) differs from $J^T$ on the domain $0 < x^T < (x^*)^{-1}$. This difference, however, will not affect the results. We wish to show in this section that, for small disturbances (to be made precise) the modified deterministic decision rule is optimal under uncertainty and that uncertainty reduces the steady-state growth rate of the system. It follows a fortiori that if the uncertainty leads to a decision rule that differs from the deterministic case, then the growth rate of the system will be reduced due to the concavity of the objective function.

As noted in the previous section, one of the difficulties associated with the optimal allocation rule under uncertainty is the choice of which outcome to steer towards the target; the other problem, of course, is the choice of the target. Define the modified deterministic decision rule (M) as follows:

(36) 

i) $x^t < (x^*)^{-1}$: $\theta^t = x^t, \gamma^t = 0; x^{t+1}(1) < x^*.$

ii) $x^t \in [(x^*)^{-1}, x^*]:$ $\theta^t = \frac{a(1 + x)x^*}{(1 + 2ax^*)}, x^{t+1}(1 - \varepsilon) < x^{t+1}(1) = x^* < x^{t+1}(1 + \varepsilon)$

$\gamma^t = x^t - \theta^t$

iii) $x^t \in [x^*, x']$: $\theta^t = 1, \gamma^t = x^t - 1; x^{t+1}(1) < x^* < x^{t+1}(1 + \varepsilon)$

where $x^* < x' = \left[\frac{1 + \varepsilon + ax^*}{ax^*}\right] < (1 + \varepsilon)x^*$

iv) $x^t > x'$: $\theta^t = 1, \gamma^t$ indeterminate, but:

$(x^*)^{-1} < x^{t+1}(1 - \varepsilon) < x^{t+1}(1 + \varepsilon) \leq x^*.$

Note that this decision rule differs from the deterministic one for $x^t \in [x^*, x']$; in this interval, all resources are fully utilized.

As we have seen for the deterministic case, the interval $[(x^*)^{-1}, x^*]$ is an important one in that it is the region aimed for, at $(t+1)$, if $x^t$ is
large. Because $x^{t+1}$ depends on the realization of $a$, it is possible that for $x^{t+1}(1 + \varepsilon) = x^*, x^{t+1}(1 - \varepsilon) < (x^*)^{-1}$; this possibility would complicate the analysis. To simplify, we assume:

\[(37) \quad 0 < \varepsilon < \left(\frac{(x^*)^2 - 1}{(x^*)^2 + 1}\right) \varepsilon[3/5, 1] \text{ since } x^* \geq 2.\]

This assumption, which still allows for sizeable $\varepsilon$, assures that $x^{t+1}(1 - \varepsilon) > (x^*)^{-1}$ whenever $x^{t+1}(1 + \varepsilon) \geq x^*$.

Assuming (37) holds, we shall show:

**Theorem 3:** For small $z$, the modified deterministic decision rule (M) is optimal. Furthermore, the steady-state growth rate under uncertainty is less than that under certainty, and is a decreasing function of $\varepsilon$ and $z$.

**Proof:**

The proof is similar to that of Theorem 1. First we show by induction that the return function, $J^t(F^t, P^t) = \text{Max} \{E[J^{t+1}(F^{t+1}, P^{t+1})]\}$ has the following properties for small $z$:

\[(38) \quad \text{i) } J^t(F^t, P^t) \text{ is a continuous concave, piecewise linear function.}\]

\[\text{ii) } J^t(F^t, P^t) = H^0(F^t, P^t); \text{ for } x^t < (x^*)^{-1}\]

\[= P^t b^t_1(1 + x^t) \text{ for } x^t \varepsilon[(x^*)^{-1}, x^*] \]

\[= P^t (b^t_2x^t + c^t_2) \text{ for } x^* < x^t \leq x^* + \delta \]

\[= H^1(F^t, P^t) \text{ for } x^t \varepsilon(x^* + \delta, x') \]

\[= P^t b^t_3(x^t + \frac{1-a}{a}) \text{ for } x^t > x',\]

where $x'$ is given by (36), and $(x^* + \delta) = \left[\frac{1 + \varepsilon + ax'}{ax'}\right] \varepsilon(x^*, x').$
Note that \((x^* + \delta)\) is defined such that \(x^{t+1}(1 + \varepsilon) \geq x^t\) for \(x^t \in [x^*, x^* + \delta], \quad \theta^t = 1, \quad \gamma^t = x^t - 1.\)

The first part of (38i) follows immediately from the properties of the terminal value function, (35), and from the linearity of the constraints and feedback rules. The second part follows from the optimality of the modified decision rule. Given \(J^{t+1}\), the modified decision rule is optimal at \(t\) if:

\[
\begin{align*}
E[\alpha J_F^{t+1} - 2J_P^{t+1}] &> 0 \quad \text{for } x^{t+1}(1) < x^* \\
E[\alpha J_F^{t+1} - 2J_P^{t+1}] &< 0 \quad \text{for } x^{t+1}(1) > x^*, \text{ and} \\
E[J_P^{t+1} - aJ_F^{t+1}] &\geq 0 \quad \text{as } x^{t+1}(1 + \varepsilon) \geq x^*, \text{ given} \\
(x^*)^{-1} &< x^{t+1}(1 - \varepsilon)
\end{align*}
\]

For \(\theta^t < \text{Min}[x^t, 1]\) it follows that \(E[J^{t+1}]\) can be increased by increasing \(\theta^t\) and decreasing \(N^t\) (decreasing \(\hat{N}^t\), increasing \(N^t\)) if \(E[\alpha J_F^{t+1} - 2J_P^{t+1}] > 0\) (if \(E[\alpha J_F^{t+1} - 2J_P^{t+1}] < 0\)). For \(\theta^t = 1\), \(E[J^{t+1}]\) can be increased by increasing \(N^t\) (if feasible—i.e., if \(\gamma^t < x^t - 1\)) if \(E[J_P^{t+1} - aJ_F^{t+1}] > 0\). Thus, if \(J^{t+1}\) obeys (39), the modified decision rule is optimal.

By assumption, \(J^T\), as given by (35), obeys (38); the modified decision rule at (T-1) will be optimal if (39) holds, i.e., if:

\[
\begin{align*}
z(1 - 2a - \varepsilon) + (1 - 2z)(1 - a + ax^*) &> 0 \quad \text{and} \\
z(1 - 2a - \varepsilon) - (1 - 2z)a(1 + x^*) &< 0.
\end{align*}
\]

Obviously, (40) holds for small \(z\); note, however, that as \(z \to (1/2)\), both relations in (40) cannot hold unless \((1 - 2a - \varepsilon) = 0\); i.e., for large \(z\) the deterministic decision rule will no longer be optimal.

Next, we must show \(J^{T-1}\) has the required property. For \(x^{T-1} < (x^*)^{-1}\) there is no problem, since \(J^{T-1}\) is concave. Substituting the optimal decision rule at (T-1) yields:
\[ J^{T-1} = \frac{p^{T-1}(1+x\ast)}{(1+2ax\ast)} [(1-z)b_1^T(1+x\ast)+zb_2^T(\frac{1-a}{a}+x\ast)-\varepsilon x\ast(b_1^T-b_2^T)], \quad x^{T-1} \varepsilon (x\ast)^{-1}, x\ast] \]
\[
= \frac{p^{T-1}}{a} [(1-z)b_1^T(1-a+ax\ast)+zb_2^T((1-a)x+a)-\varepsilon z(b_1^T-b_2^T)], \quad x^{T-1} \varepsilon (x\ast,x\ast+\delta)
\]
\[
= p^{T-1}b_1^T[\frac{1-a}{a} + x],
\]
\[x^{T-1} > (x\ast + \delta)\]

which has the form given by (38).

Note that if \( J^{t+1} \) has the form given by (38), and if \( M \) is optimal at \( t \), then \( b_1^t, b_2^t, c_2^t, \) and \( b_3^t \) are given by:

\[ b_1^t = \frac{(1-z)(1+x\ast)b_1^{t+1} + zb_3^{t+1} \frac{1-a}{a} + x\ast)}{(1 + 2ax\ast)} - \varepsilon x\ast(b_1^t-b_3^t) \]

\[ b_2^t = (1-z)b_1^{t+1} + z \cdot b_3^{t+1} \frac{1-a}{a} \]

\[ c_2^t = [(1-z)\frac{(1-a)}{a}b_1^{t+1} + zb_3^{t+1} - \varepsilon z(b_1^{t+1} - b_3^{t+1})] \]

\[ b_3^t = b_1^{t+1} \]

Thus far, we have shown \( M \) to be optimal at \( (T-1) \) for small \( z \); and that if \( M \) is optimal at \( t \), and \( J^{t+1} \) has the required form, then so will \( J^t \). Given that \( J^{t+1} \) has the required form, \( M \) will be optimal at \( t \) if:

\[ z(b_1^{t+1} - b_3^{t+1}) \frac{(1 - 2a - \varepsilon)}{a} + (1 - 2z)b_1^{t+1} \frac{(1 - 2a)}{a} > 0, \quad \text{and} \]

\[ z(b_1^{t+1} - b_3^{t+1}) \frac{(1 - 2a - \varepsilon)}{a} + (1 - 2z)(b_2^{t+1} - 2ac_2^{t+1}) < 0, \]

where:

\[ b_2^{t+1} - 2ac_2^{t+1} = -(1-z)(1-2a)b_1^{t+2} + \frac{zb_3^{t+2}}{a}(1-2a)(1+a) + 2\varepsilon z(b_1^{t+2} - b_3^{t+2}) \]
From (43'), it is clear that, for small \( z \), \( (b_{2}^{t+1} - 2ac_{2}^{t+1}) < 0 \); and hence, for small \( z \), (43) holds.

Therefore, since \( M \) is optimal at (T-1) for small \( z \), \( J_{T-1}^{T} \) has the required form; hence, from (43) and (43'), \( M \) is optimal at (T-2) if \( z \) is small.

Consequently, \( J_{T-2}^{T} \) has the required form and, by induction, the decision rule \( M \) is optimal and \( J_{t}^{T} \) has the required form for all \( t \), provided \( z \) is small.

This completes the proof of the first part of Theorem 3.

It is clear from (42) that it is sufficient for the description of the steady-state growth rate to know the behavior of \( b_{1}^{t} \). In particular, the expected steady-state rate of growth \( \phi \) must satisfy:

\[
(44) \quad b_{1}^{t} = \phi b_{1}^{t+1}
\]

Combining (42) and (44), one has a second-order linear difference equation with constant coefficients. After some algebraic manipulation involving (27) and (28), this equation becomes:

\[
(45) \quad \phi^2 = (1-z)\phi(x^*-1) + z(x^*-1)^2 - \frac{\varepsilon xx^*(x^*-1)(\phi-1)}{(x^*+1)}
\]

For \( z = \varepsilon = 0 \), \( \phi = (x^*-1) \), the deterministic growth rate.

Rearranging (45):

\[
(46) \quad [\phi-(x^*-1)][\phi+z(x^*-1)] + \frac{\varepsilon xx^*(x^*-1)}{(x^*+1)}(\phi-1) = 0.
\]

From (46) it is obvious that for \( \varepsilon \) satisfying (37) and small \( z \),

\[
(47) \quad 1 < \phi < x^* - 1.
\]

thus proving the second part of the theorem.
To show the second assertion, write (45) in the form:

(48) \[ F(\phi, z, \varepsilon) = 0. \]

Then, at \( F = 0 \), one has

\[
\frac{\partial F}{\partial \phi} = 2\phi - (1-z)(x^*-1) + \frac{z\phi(x^*-1)}{(x^*+1)} > 0
\]

\[
\frac{\partial F}{\partial z} = (x^*-1)[\phi - (x^*-1) + \frac{z\phi(x^*-1)}{(x^*+1)}] = -\frac{\phi - (x^*-1)}{z} > 0
\]

\[
\frac{\partial F}{\partial \varepsilon} = \frac{z\phi(x^*-1)}{(x^*+1)}(\phi-1) > 0
\]

Therefore \( \frac{\partial \phi}{\partial z}, \frac{\partial \phi}{\partial \varepsilon} < 0 \), proving the remainder of the theorem.

As a by-product of Theorem 3, one can compute a bound on how large \( z \) can become before the policy rule \( M \) is no longer optimal. As stated before that rule is optimal at time \( t-1 \) if

(48) \[ E[\alpha^t_F - 2J^t_p] > 0 \quad \text{for} \quad x^t(1 - \varepsilon) < x^t(1) < x^* < x^t(1 + \varepsilon) \]

(49) \[ E[\alpha^t_F - 2J^t_p] < 0 \quad \text{for} \quad x^t(1 - \varepsilon) < x^* < x^t(1) < x^* + \delta \]

By analogy with Theorem 2, the \( z \) values which make (48) hold with equality we shall denote \( z^*_t \); those making (49) hold with equality, \( \tilde{z}_t \).

From (43), the following implicit equation defines \( z^*_t \)

(50) \[ [(1 - z^*_t)b_{1}^{t+1} - b_{1}^{t+2}z](1 - 2a) - \varepsilon z^*_t(b_{1}^{t+1} - b_{1}^{t+2}) = 0. \]

From (43) and (43') \( \tilde{z}_t \) must satisfy
\[ (51) \quad \tilde{z}_t (1 - 2a - \varepsilon)(b_1^{t+1} - b_1^{t+2}) + 2\varepsilon \tilde{z}_t (1 - \tilde{z}_t) (b_1^{t+1} - b_1^{t+2}) \]

\[ - (1 - 2\tilde{z}_t) (1 - 2a)[(1 - \tilde{z}_t) b_1^{t+1} - \tilde{z}_t b_1^{t+2} (1 + a) a] = 0 \]

Although both \( z_t^x \) and the \( \tilde{z}_t \) fluctuate over time, they rapidly approach their asymptotic limits, found by substituting \( b_1^t = \varphi b_1^{t+1} \) in (50) and (51).

To illustrate the various aspects of Theorem 3, let \( \lambda = 1.05 \), \( \varepsilon = .07 \) and \( z = .2 \). Then \( \varphi = 1.049 \), not a very large growth less initially but one that compounds with time. For all times \( t \), (48) and (49) hold. In this case, \( \tilde{z}_{\infty} = .24 \) is the only steady-state root of either (50) or (51) which is below one-half.

VI. Conclusion

The results of this paper have shown the impact on resource allocation and expected growth in the 2-sector von Neumann model under uncertainty. These results, we believe, have important implications for future research. First, the negative impact on expected growth of increases in risk suggests as yet untested hypotheses about the recently observed growth slowdown in several planned economies. That increased risk now faces such economies seems a plausible description of the economic environment in which their planners operate. Second, the serious difficulties encountered in computing the optimal plan under all parameter values highlights once more the role that satisficing concepts must play in practical applications. As Radner has forcefully suggested [11, 12], bounded rational solutions may be the best that can be hoped for in general planning models. Indeed, one can interpret the modified deterministic decision rule in our model as such a bounded rational solution. In cases where this policy can be shown not to be optimal, one can also obtain a bound on the growth loss involved in using it. Such
a calculation serves as a further guide to the possible gains involved when one tries to compute a better decision rule.

Several extensions of the model seem desirable. The extension to many sectors, besides the obvious greater generality, would allow for the investigation of the phenomenon of forced substitution, as emphasized in the work of Kornai [5, 6]. Although forced substitution plays no role in the model of this paper, it would also appear to be a factor in growth loss due to uncertainty. Again, Kornai and Simonovits [7] have shown in the context of deterministic von Neumann control models that an increase in buffer stock norms causes growth loss. The results of our model make us suspect that quite the opposite may be true under uncertainty. Finally, the uncertainty dealt with in the present model is exclusively exogenous, whereas it appears that a substantial amount of endogenous uncertainty is actually generated by plan implementation. It would be useful to measure the extent of this uncertainty, since it can in principle be reduced through institutional change. The authors intend to deal with these issues in the future.
Footnotes

* Many of the ideas for this paper were developed while one of the authors was Visiting Research Scholar at the Institute for International Economic Studies Stockholm. He is indebted to the Institute for their generous support during this time and to János Kornai for numerous enlightening conversations. Naturally, the views expressed in the paper are solely those of the authors.

1/ $J^t$ consists of linear segments for $x^t < x_L$, the number of which depends on the horizon. Of course, $J^t$ is not differentiable at its vertices; for any vertex, $x_i$, of $J^t$, define $J_{P_i}^t = \lim_{x^t(x_i^t)} (\frac{\partial J^t}{\partial P_i})$, i.e., as the right-hand derivative.

2/ Actually, $x_U^t$ does not converge monotonically, but oscillates around $x^*; x_U^t > x^* \rightarrow x_U^{t-1} < x^*$. However, $x_U^t > x^* \rightarrow x_U^{t-1} > x^{t-2} > x^*$, and $x_U^t > x^* \rightarrow x_U^t < x_U^{t-2} < x^*$. Thus, $|x_U^{t-2} - x^*| < |x_U^t - x^*|$, so the alternating sequences converge monotonically.

3/ A continuous random variable with mean one leads to qualitatively similar results. Hereafter, the time subscript on $\alpha$ will be suppressed.

4/ While specifying a different terminal value for $P^{T+1}$ will alter this conclusion, the effect will not be appreciable in earlier periods.

5/ The complete solution for $T-2$ is available from the authors upon request.

6/ For our purposes, the parts of the return function denoted by $H^0$ and $H^1$ play no role and can therefore be left unspecified. Of course, $H^0$ and $H^1$ must be such as to preserve the continuity and concavity of $J^t$. 
References


