On the Relevance of Fractional Gaussian Processes for Analysing Financial Markets

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Abstract

In recent years, the field of Fractional Brownian motion, Fractional Gaussian noise and long-range dependent processes has gained growing interest. Fractional Brownian motion is of great interest for example in telecommunications, hydrology and the generation of artificial landscapes. In fact, Fractional Brownian motion is a basic continuous process through which we show that it is neither a semimartingale nor a Markov process. In this work, we will focus on the path properties of Fractional Brownian motion and will try to check the absence of the property of a semimartingale. The concept of volatility will be dealt with in this work as a phenomenon in finance. Moreover, some statistical method like R/S analysis will be presented. By using these statistical tools we examine the volatility of shares and we demonstrate empirically that there are in fact shares which exhibit a fractal structure different from that of Brownian motion.

Keywords:  Fractional Brownian motion, Fractional Gaussian noise, semimartingale, volatility.

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Introduction

Fractional Brownian motion was first studied by Kolmogorov [26]. He did not use the name of Fractional Brownian motion, but he called it the Wiener spiral process. Mandelbrot and Van Ness called this processes Fractional Brownian motion. They introduced the Hurst-index $H$ with range $(0, 1)$ [9].

In general, self-similar processes as Fractional Brownian motion (which we will study later), are being applied in economics and natural sciences, such as hydrology. There are some applications of Fractional Brownian motion in telecommunications and network traffic as well. In economics Fractional Brownian motion was suggested as a model, see [9]. Fractional Brownian models can be used to model the volatility of asset prices, as suggested by Djehiche and Eddahbi [23].

Section 1 contains basic facts on Fractional Brownian motion and on its sample path properties. Its $p$-variation will be discussed too. Section 2 discusses why Fractional Brownian motion fails to be a semimartingale. In section 3 we present one way to make sense of integration with respect to Fractional Brownian motion and the last section is above all dedicated to the statistical approach.

1 Basic facts on Fractional Brownian motion

We say that a stochastic process $X_t, t \in T$, with Gaussian finite dimensional distributions is a Gaussian process. Furthermore the finite dimensional distributions are equally determined by covariance matrix entries which are given by the pair covariance function $R(s, t) = R[X_s, X_t] = E[(X_t - E[X_t])(X_s - E[X_s])]$, where $E[X_t]$ denotes the expectation of the random variable $X_t$.

Definition 1.1. a) We say that a random process $X = (X_t)_{t \geq 0}$ with state space $\mathbb{R}^d$ is self-similar or satisfies the property of (statistical) self similarity if for each $a > 0$ there exists $b > 0$ such that

$$(X_{at}, t \geq 0) = (bX_t, t \geq 0) \quad \text{in law.}$$

In other words changes in the time scale $(t \rightarrow at)$ produce the same results as changes in the rescaling of state space $(x \rightarrow bx)$ [7].

b) In the case of general stable processes, instead of (1), we have the property

$$(X_{at}, t \geq 0) = (a^H X_t + tD_a, t \geq 0) \quad \text{in law.}$$

This means that a change in the time scale $(t \rightarrow at)$ produces the same results as a change in the rescaling of state space $(x \rightarrow a^H x)$ and a subsequent translation defined by the vector $tD_a, t \geq 0$.

Now it is reasonable to introduce the following definition.

Definition 1.2. If $b = a^H$ in Definition [7] for each $a > 0$, then we call $X = (X_t)_{t \geq 0}$ a self-similar process with Hurst exponent $H$ or we say that this process has the property of statistical self-similarity with Hurst exponent $H$ [7].

According to a result of probability theory the characteristic function

$$\phi(\theta) = E[e^{i\theta X}]$$

of a stable random variable $X$ has the following representation:

$$\phi(\theta) = \left\{ \begin{array}{ll}
\exp \left\{ i\mu \theta - \sigma^\alpha |\theta|^\alpha (1 - i\beta (\text{Sgn}\theta) \tan \frac{\pi \alpha}{2}) \right\} & \text{if } \alpha \neq 1, \\
\exp \left\{ i\mu \theta - \sigma |\theta| (1 + i\beta \frac{\ln |\theta|}{2}) \right\} & \text{if } \alpha = 1,
\end{array} \right.$$
where $0 < \alpha \leq 2$, $|\beta| \leq 1$, $\sigma > 0$ and $\mu \in \mathbb{R}$.

The parameters $(\alpha, \beta, \sigma, \mu)$ have the following meaning:

$\alpha$ is the stability exponent or the characteristic parameter,
$\beta$ is the skewness parameter of the distribution density,
$\sigma$ is the scale parameter and $\mu$ is the location parameter.

If $\alpha = 0$, then by (2) we obtain

$$\varphi(\theta) = e^{i\mu \theta - \sigma^2 \theta^2} = e^{i\mu \theta - \theta^2/(2\sigma^2)},$$

which shows that $\varphi(\theta)$ is the characteristic function of the normal distribution $N(\mu, 2\sigma^2)$.

**Definition 1.3.** Fractional Brownian motion of Hurst-index $H \in (0, 1)$ is a zero-mean Gaussian process $Z^H$ with covariance

$$R^H(s, t) = \frac{1}{2}(s^{2H} + t^{2H} - |t - s|^{2H}),$$

where $s, t \geq 0$. For the sake of simplicity we assume that $Z^H(0) = 0$.

Especially when $H = \frac{1}{2}$, we obtain standard Brownian motion.

We observe that Fractional Brownian motion is the only Gaussian self-similar process with stationary increments. This is why it is interesting to study this process.

The Hurst-index $H$ stands not only for the sign of the sample but also for the regularity of the sample paths. If $H > \frac{1}{2}$, then the correlations for the increments are positive and if $H < \frac{1}{2}$, the increments are negatively correlated.

**Lemma 1.1.** If $H \in (0, 1)$ then the function $R^H$ in (4) is a non-negative definite function [4].

**Remark.** According to Bochner’s theorem [13] any non-negative definite function defines a unique zero mean Gaussian process using the characteristic function (3), see the definition of Gaussian process in [13]. Thus we can define Fractional Brownian motion as above, to be the zero mean Gaussian process with covariance the function $R^H, H \in (0, 1)$.

We take an example when $H = 1$

$$E[(X_t - tX_1)^2] = E[X_t^2] - 2tE[X_tX_1] + t^2E[X_1^2]$$

$$= (t^2 - 2t \cdot t + t^2)E[X_1^2]$$

$$= 0$$

We conclude that for a fixed path $\omega$, $X_t(\omega) = tX_1(\omega)$, which represents a straight line through the origin with slope $X_1(\omega)$. Thus we exclude the case $H = 1$.

**Definition 1.4.** For any arbitrary process $(\xi_n)_{n \in \mathbb{N}}$ let us introduce the autocorrelation function $\rho(n) = E[\xi_0, \xi_n]$. If any $(\xi_n)_{n \in \mathbb{N}}$ $\rho$ decays so slowly then

$$\sum_{n=1}^{\infty} \rho(n) = \infty$$

we say that $(\xi_n)_{n \in \mathbb{N}}$ exhibits long-range dependence. If $\rho$ decays exponentially, i.e. $\rho(n) \sim r^n$ as $n$ tends to infinity, then the stationary sequence $(\xi_n)_{n \in \mathbb{N}}$ exhibits short-range dependence.

There are many definitions for long-range dependence. For more details consider [5].

Defining a process through its finite dimensional distribution does not always give a clear picture of its underlying structure. The increment process, called Fractional Gaussian noise is introduced in the following subsection.
1.1 Fractional Gaussian noise

Brownian motion $B = (B_t)_{t \geq 0}$ is regarded in many domains of applied probability as a way to obtain white noise

$$\beta_n = B_n - B_{n-1}, \quad n \geq 1.$$  

This means that we obtain a sequence $\beta = (\beta_n)_{n \geq 0}$ of independent uniformly distributed random Gaussian variables with $E(\beta_n) = 0$ and $E(\beta_n^2) = 1$.

In the same way, we obtain the following for Fractional Brownian motion $Z^H$.

If $(Y_n)_{n \in \mathbb{N}}$ is the stationary sequence with

$$Y_n = Z_{n+1}^H - Z_n^H$$

where $Z^H$ is Fractional Brownian motion with Hurst-index $H$ then we call $(Y_n)_{n \in \mathbb{N}}$ Fractional Gaussian Noise with Hurst-index $H$.

By using the formula for the covariance function of Fractional Brownian motion \(^{[4]}\) we obtain that the covariance function for Fractional Gaussian noise $\rho_H(n) = \text{Cov}(Y_k, Y_{k+n})$ is

$$\rho_H(n) = \frac{1}{2} \left( |n+1|^{2H} - 2|n|^{2H} + |n-1|^{2H} \right).$$

For large $n$ this can be interpreted as the discrete version of the second derivative, as the following shows

$$\frac{(|n+1|^{2H} - |n|^{2H})}{(n+1)-(n-1)} - \frac{(|n|^{2H} - |n-1|^{2H})}{(n-(n-1))} = \frac{2H(|n+1|^{2H-1} - |n-1|^{2H-1})}{(n+1)-(n-1)}.$$  

Hence, the autocorrelation function $\rho = \rho_H$ of Fractional Gaussian noise with $H \neq \frac{1}{2}$ satisfies

$$\rho(n) \sim H(2H-1)n^{2H-2}$$  

as $n$ tends to infinity \(^{[4]}\) and \(^{[7]}\). We see that for $H = \frac{1}{2}$, then we find $\rho_H(n) = 0$ which is the case of a Gaussian sequence of independent random variables. But when $H \neq \frac{1}{2}$, then the covariance decreases slowly as $n$ increases interpreted as a long memory.

We obtain, as mentioned above, that if $H > \frac{1}{2}$, then the increments of the corresponding Fractional Brownian motion are positively correlated and exhibit the long range dependence property. And if $H < \frac{1}{2}$, then the increments of the corresponding Fractional Brownian motion are negatively correlated.

**Proposition 1.2** (Chain rule). For any $\sigma$-fields $\mathcal{G}, \mathcal{H},$ and $\mathcal{F}_1, \mathcal{F}_2, \ldots$, these conditions are equivalent:

(i) $\mathcal{H} \perp_\mathcal{G} (\mathcal{F}_1, \mathcal{F}_2, \ldots)$;

(ii) $\mathcal{H} \perp_\mathcal{G} \mathcal{F}_1, \ldots, \mathcal{F}_n, \mathcal{F}_{n+1}$, $n \geq 0$,

where the symbol $\perp$ stands for the independence. For proof see \(^{[6]}\).

**Proposition 1.3** (Gaussian Markov processes). Let $X$ be a Gaussian process on some index set $T \subset \mathbb{R}$, and define $R(s, t) = R[X_s, X_t]$ to be a covariance. Then $X$ is a Markov process if and only if

$$R(s, u) = \frac{R(s, t)R(t, u)}{R(t, t)}, \quad s \leq t \leq u \text{ in } T,$$

where $0 \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0$ \(^{[6]}\).
Lemma 1.4. We assume that $E[X_t] = 0$; we fix arbitrary times $s < t < u$ in an arbitrary closed interval $T \in \mathbb{R}^+$, and we choose $X'_u = X_u - aX_t$, for some $a \in \mathbb{R}$. Then $X'_u$ and $X_t$ are independent if and only if $a = \frac{R(t,u)}{R(t,t)}$.

Proof. Applying the computation rules for covariances we find that $X'_u$ and $X_t$ are uncorrelated if and only if $a = \frac{R(t,u)}{R(t,t)}$.

$$R[X'_u, X_t] = R[X_u, X_t] - aR[X_t, X_t] = R[X_u, X_t] - \frac{R(t,u)}{R(t,t)} R[X_t, X_t] = 0.$$  

We recall that $X'_u$ is Gaussian. For the Gaussian random variables $X'_u$ and $X_t$, which are uncorrelated, we automatically obtain that $X'_u$ and $X_t$ are independent.

Proof. (Proof of Proposition 1.3) We assume that $X$ is Markov. Let $s < t < u$ be arbitrary. Then $X_s$ and $X_u$ are independent given $X_t$, and so $X_s$ and $X'_u$ are independent given $X_t$, since $X_t$ is also independent of $X'_u$ by the choice of $a$ in the lemma above. By Proposition 1.2 we obtain that $X_t$ and $X'_u$ are independent. Hence, $R(s,u) = aR(s,t)$. We obtain (6) as we insert the expression $a = \frac{R(t,u)}{R(t,t)}$.

Conversely, (6) implies $X_s \perp X'_u$ for all $s \leq t$. By checking that $R(X_s, X'_u) = 0$, we obtain that $X_s$ and $X'_u$ are independent given $X_t$ by Lemma 11.1 in [6]. In other words, $\mathcal{F}_s$ and $\mathcal{F}_u$ are independent given $\mathcal{F}_t$ where $\mathcal{F}_t = \sigma\{X_s; s \leq t\}$. By Proposition 1.2 we obtain

$$P[\mathcal{F}_u | \mathcal{F}_t, \mathcal{F}_s] = P[\mathcal{F}_u | \mathcal{F}_t]$$

reintroducing our original notation

$$P[X'_u | X_t, X_s] = P[X'_u | X_t]$$

for arbitrary $s < t < u$, taking expectation

$$E[X'_u | X_t, X_s] = E[X'_u | X_t],$$

which is a Markov property.

Proposition 1.5. Fractional Brownian motion with Hurst-index $H$ is a Markov process if and only if $H = \frac{1}{2}$.

Proof. By Proposition 1.3 we know that a Gaussian process with covariance $R$ is Markovian if and only if its covariance function satisfies (6). We are going to show that the covariance function (4) satisfies the condition of Proposition 1.3 if and only if $H = \frac{1}{2}$.

We begin by demonstrating that $H = \frac{1}{2}$ is a sufficient condition for $Z^H$ to be Markovian. For $H = \frac{1}{2}$, the function (4) becomes

$$R[Z^\frac{1}{2}(s), Z^\frac{1}{2}(t)] = \frac{1}{2}(s + t - |s - t|)$$

$$= \begin{cases} 
\frac{1}{2}(s + t - (s - t)) = t & \text{if } s > t \\
\frac{1}{2}(s + t + (s - t)) = s & \text{if } s \leq t
\end{cases}$$

In other words, for $H = \frac{1}{2}$ we obtain a Brownian motion. Introducing this particular covariance for $s \leq t \leq u$ into the above equation reveals that $R(s,t) = s, R(t,t) = t$ and
$R(t,u) = t$. By substituting the covariance function in (7) and using the particular expression (4) we find for $s \leq t \leq u$:

$$\frac{R(s,t)R(t,u)}{R(t,t)} = \frac{st}{t} = s = R(s,u).$$

According to Proposition 1.3 Fractional Brownian motion is Markovian.

Conversely, let us assume that $H \neq 0$. Then we can rewrite (6) as $R(t,t)R(s,u) - R(s,t)R(t,u) = 0$, where $s \leq t \leq u$. For our particular case let us show that $H = \frac{1}{2}$ is a necessary condition.

We have for all $0 \leq s \leq t \leq u < \infty$

$$\frac{1}{4}(t^{2H} + t^{2H} - |t-t|^{2H})(s^{2H} + u^{2H} - |s-u|^{2H}) = \frac{1}{4}(t^{2H} + t^{2H} - |s-t|^{2H})(t^{2H} + u^{2H} - |t-u|^{2H})$$

$$= 2t^{2H}(s^{2H} + u^{2H} - (u-s)^{2H}) - (t^{2H} + t^{2H} - (t-s)^{2H} - (t^{2H} + u^{2H} - (u-t)^{2H})$$

$$= 2t^{2H} + s^{2H} - 2t^{2H}u^{2H} - 2t^{2H}(u-s)^{2H} - s^{2H}u^{2H} + s^{2H}(u-t)^{2H} - t^{4H} +$$

$$t^{2H}(u-t)^{2H} + t^{2H}(t-s)^{2H} + u^{2H}(t-s)^{2H} - (t-s)^{2H}(u-t)^{2H} \equiv 0,$$

which is only true for $H = \frac{1}{2}$. This indicates that our assumption is false, thus $H = \frac{1}{2}$ and the only if side is true.

\[ \square \]

1.2 Regularity of sample paths of Fractional Brownian motion

1.2.1 Non-differentiability of Fractional Brownian motion

**Definition 1.5.** A stochastic process $Z^H = (Z^H_t)_{t \in [0,1]}$ is $\beta$-Hölder continuous if there exists a finite random variable $K$ such that

$$\sup_{s,t \in [0,1]; s \neq t} \frac{|Z^H_t - Z^H_s|}{|t-s|^\beta} \leq K$$

for almost all trajectories.

**Remark.** If $(B_t)_{t \geq 0}$ is a d-dimensional Brownian motion, then almost every $\omega$ in the underlying probability space $\Omega$ has the following property: For each $\beta > \frac{1}{2}$, the path $t \rightarrow B_t(\omega)$ is nowhere Hölder-continuous of order $\beta$. And then it is almost sure that the path is nowhere differentiable [13].

In this paragraph we adapt to [4].

**Proposition 1.6.** Fractional Brownian motion with Hurst-index $H$ denoted by $Z^H$ admits a version with $\beta$-Hölder continuous sample paths if $\beta < H$. If $\beta \geq H$, then Fractional Brownian motion is almost surely not $\beta$-Hölder continuous on any time interval.

**Proof.** Let $\beta < H$ and $n \in \mathbb{N}$. Then, by self-similarity and stationarity of the increments we have

$$E[|Z^H_t - Z^H_s|^n] = E[|Z^H_t - Z^H_{s+\gamma_n}|^n] = E[|t-s|^H Z^H_{s+\gamma_n}] = |t-s|^n H_{\gamma_n},$$

where $\gamma_n$ is the $n^{th}$ absolute moment of a standard normal random variable. The Kolmogorov-Chentsov criterion [13] implies that the sample paths of $Z^H$ are almost surely $H$-Hölder continuous of any order less than one. By stationarity of the increments it is enough to consider the point $t = 0$.

Now, we will show that $Z$ cannot be $H$-Hölder continuous for $\beta > H$ at any point $t \geq 0$.

We assume that $\beta \geq H$ then $\beta = H + \varepsilon$, where $\varepsilon \geq 0$. Then

$$\frac{Z_t - Z_0}{t^\beta} = \frac{Z_t - Z_0}{t^{H+\varepsilon}} = \frac{Z_t - Z_0}{t^H \cdot t^\varepsilon}.$$
Now we take the supremum limit, we obtain

$$\limsup_{t \to 0} \frac{Z_t - Z_0}{t^\beta} = \limsup_{t \to 0} \frac{Z_t - Z_0}{t^H} \lim_{t \to 0} t^{-\varepsilon}$$

(8)

As a consequence of results in [8] we obtain that (8) becomes

$$\limsup_{t \to 0} 1 \sqrt{\ln \frac{1}{t} t^{-\varepsilon}}$$

which diverges to infinity as $t \to 0$ since both factors diverge. Then, there is no finite $K$ such that $\frac{Z_t - Z_0}{t^\beta} \leq K$ and $Z$ cannot be $H$-Hölder continuous for $\beta \geq H$.

The non-differentiability is not connected to Fractional Brownian motion being a Gaussian process but rather being self similar.

**Definition 1.6.** The increments of a random function $X(t, \omega)$ defined for $-\infty < t < \infty$ are said to be self-similar with the exponent $H \geq 0$ if, for any $h > 0$ and for any $t_0$,

$$\{X(t_0 + \tau, \omega) - X(t_0, \omega)\} \stackrel{\Delta}{=} \{h^{-H}[X(t_0 + h\tau, \omega) - X(t_0, \omega)]\}.$$ 

**Remark.** The notation $\{X(t, \omega)\} \stackrel{\Delta}{=} \{Y(t, \omega)\}$ means that the two random functions $\{X(t, \omega)\}$ and $\{Y(t, \omega)\}$ have the same finite joint distribution functions.

**Proposition 1.7.** Almost all sample paths of Fractional Brownian motion $Z(t, \omega)$ are not differentiable for any $t$. 

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Proof. Let us now study the limit supremum, in fact
\[
\limsup_{t \to t_0} \left| \frac{Z^H(t, \omega) - Z^H(t_0, \omega)}{t - t_0} \right| = \infty
\]
with probability one, that the limit supremum of the differential quotient diverges as \( t \to t_0 \).

Assuming \( Z^H(0) = 0 \), the identity in Definition 1.6 yields
\[
\frac{Z^H(t - t_0, \omega)}{t - t_0} = \frac{Z^H(t, \omega) - Z^H(t_0, \omega)}{t - t_0} \Delta \approx (t - t_0)^{H-1} \{ Z^H \left( \frac{t_0}{t - t_0} + 1, \omega \right) - Z^H \left( \frac{t_0}{t - t_0}, \omega \right) \} + (t - t_0)^{H-1} Z^H(1, \omega).
\]

Define the events
\[
A(t, \omega) = \left\{ \sup_{0 \leq s \leq t} \left| \frac{Z^H(s, \omega)}{s} \right| > d \right\},
\]
for arbitrary large \( d > 0 \). For any sequence such that \( t_n \to 0, n \in \mathbb{N} \), we have
\[
A(t_n, \omega) \supset A(t_{n+1}, \omega);
\]
thus by continuity of measures
\[
P \left[ \lim_{n \to \infty} A(t_n) \right] = \lim_{n \to \infty} P[A(t_n)],
\]
and
\[
P[A(t_n)] \geq P \left[ \frac{Z^H(t_n)}{t_n} > d \right] = P \left[ |Z^H(1)| > t_n^{1-H} d \right],
\]
which tends to 1 as \( n \to \infty \) [9]. We obtain the last equality by taking in the normal distribution property. \( \square \)

1.3 The \( p \)-variation of Fractional Brownian motion

We are going to introduce another notion of path regularity, namely the \( p \)-variation. Referring to equation (4), it follows that the variance \( (\sigma^H)^2 \) of the increments \( \Delta Z^H = Z^H(t + u) - Z^H(t) \) of \( Z^H \) at times \( t, t + u \) is given by:
\[
\sigma^H(u)^2 := E[\Delta Z^H^2] - E[\Delta Z^H]^2
\]
where \( E[\Delta Z^H]^2 = 0 \) by Definition 1.3. Now by direct calculation of \( E[\Delta Z^H^2] \) and using (4) we obtain
\[
E \left[ (Z^H(t + u))^2 \right] - 2E \left[ (Z^H(t + u)(Z^H(t)) \right] + E \left[ (Z^H(t))^2 \right] \n\]
\[
= \frac{1}{2} ((t + u)^{2H} + (t + u)^{2H}) - \frac{2}{2} ((t + u)^{2H} + t^{2H} - |u|^{2H}) + \frac{1}{2} (t^{2H} + t^{2H}) \n\]
\[
= \frac{1}{2} 2u^{2H} = u^{2H}.
\]
Thus \( Z^H \) has stationary increments [10].
Consider partitions \( \pi := \{t_k : 0 = t_0 < t_1 < \ldots < t_n = 1 \} \) of the interval \([0,1]\). Let \( |\pi| := \max_{t_k \in \pi} \Delta t_k \) where \( \Delta t_k := t_k - t_{k-1} \). Let \( f \) be a function on the interval \([0,1]\). Then for \( p \in [1,\infty) \)

\[
v_p(f; \pi) := \sum_{t_k \in \pi} |\Delta f(t_k)|^p
\]

where \( \Delta f(t_k) := f(t_k) - f(t_{k-1}) \) is called the \( p \)-variation of \( f \) along the partition \( \pi \).

**Definition 1.7.** Let \( f \) be a function on the interval \([0,1]\). If

\[
v_p^0 := \lim_{|\pi| \to 0} v_p(f; \pi)
\]

exists we say that \( f \) has a finite \( p \)-variation. If

\[
v_p(f) := \sup_{\pi} v_p(f; \pi)
\]

is finite then \( f \) has bounded \( p \)-variation.

When \( p = 1 \) the bounded 1-variation \( v_1 \) is the usual bounded variation. And when \( p = 2 \) the finite 2-variation \( v_2^0 \) coincides with the classical notion of quadratic variation in martingale theory.

**Definition 1.8.** The Banach space \( \Psi_p \) is the set of functions of bounded \( p \)-variation equipped with the norm

\[
\|f\|_p := \|f\|_p + \|f\|_\infty,
\]

where \( \|f\|_p := V_p(f)^{\frac{1}{p}} \) and \( \|f\|_\infty := \sup_{t \in [0,1]} |f(t)| \).

**Remark.** The function in the definition above is Hölder continuous and the interval is compact.

**Lemma 1.8.** Let \( p \in [1,\infty) \) and let \( f \) be a function over the interval \([0,1]\). Then \( f \) has bounded \( p \)-variation if and only if

\[
f = g \circ h
\]

where \( h \) is a bounded non-negative increasing function on \([0,1]\) and \( g \) is a \( 1/p \)-Hölder continuous function defined on \([h(0),h(1)]\).

**Proof.** Consider the if part. By taking \( h \) to be the identity function and supposing \( f \) to be \( 1/p \)-Hölder continuous with Hölder constant \( K \) we find that

\[
\sum_{t_k \in \pi} |\Delta f(t_k)|^p \leq \sum_{t_k \in \pi} |K| \Delta t_k |^{\frac{1}{p}}|^p \leq K^p \sum_{t_k \in \pi} |\Delta t_k |^{\frac{1}{p}}^p = K^p,
\]

for any partition \( \pi \) of the interval \([0,1]\). The \( \Psi_p \) is the collection of functions with finite \( p \)-variation. And the estimate gives that \( f \)'s \( p \)-variation along any given partition is bounded by \( K^p \). Hence, it must have finite \( p \)-variation. So \( f \in \Psi_p \).

For the only if part suppose that \( f \in \Psi_p \). Let \( h(x) \) be the \( p \)-variation of \( f \) on \([0,x] \subset [0,1]\). Then \( h \) is a bounded increasing function. Since the \( p \)-variation, where \( p \in [1,\infty) \), is subadditive with respect to intervals we can write

\[
|h(x) - h(y)| \leq |h(x) - h(y)|^p.
\]

Now define \( g \) on \( \{h(x) : x \in [0,1]\} \) by \( g(h(x)) := f(x) \) and extend it to \([h(0),h(1)]\) by linearity. We conclude that \( g \) is \( 1/p \)-Hölder continuous. \( \square \)
Lemma 1.9. Set \( \pi_n := \{ t_k = \frac{k}{n} : k = 1, \ldots, n \} \) and let \( Z^H \) be Fractional Brownian motion with Hurst-index \( H \). Let \( \gamma_p \) denote the \( p \)th absolute moment of a standard normal random variable. Then

\[
\lim_{n \to \infty} \nu_p(Z^H; \pi_n) = \begin{cases} 
\infty & \text{if } p < \frac{1}{H} \\
\gamma_p & \text{if } p = \frac{1}{H} \\
0 & \text{if } p > \frac{1}{H}
\end{cases}
\]

For a proof see [4].

When we talk about Fractional Brownian motion with Hurst-index \( H \) then the critical value for \( p \)-variation is \( 1/H \).

Proposition 1.10. Let \( Z^H \) be Fractional Brownian motion with Hurst-index \( H \). Then \( \nu^0_p(Z^H) = 0 \) almost surely if \( p > \frac{1}{H} \). For \( p < \frac{1}{H} \) we have \( \nu_p(Z^H) = \infty \) and \( \nu^0_p(Z^H) \) does not exist. Moreover \( \nu(Z^H) = \frac{1}{H} \).

Proof. By Proposition 1.6 Fractional Brownian motion \( Z^H \) is Hölder continuous for all \( \beta < H \). Let \( K \) stand for the Hölder constant, moreover let \( p > \frac{1}{H} \) and \( \pi \) be a partition of \([0,1]\). Then we find

\[
\sum_{t_k \in \pi} |\Delta Z_{t_k}|^p \leq \sum_{t_k \in \pi} ||K|\Delta Z_{t_k}^\beta|^p = K^p \sum_{t_k \in \pi} |\Delta Z_{t_k}|^\beta p = K^p \sum_{t_k \in \pi} \frac{|\pi|}{|\pi_\pi|} |\Delta Z_{t_k}|^\beta p - 1 \leq K^p |\pi| \sum_{t_k \in \pi} |\Delta Z_{t_k}|^\beta p - 1
\]

where \( |\Delta Z_{t_k}| \leq 1 \), almost surely for any \( \beta < H \). Letting \(|\pi|\) tend to zero we see that \( \nu^0_p = 0 \). Suppose then that \( p < 1/H \). Then by Lemma 1.9 we can choose a subsequence \( (\pi'_n)_n \subseteq N \) of the sequence of equidistant partitions \( (\pi_n)_n \subseteq N \) such that \( v_{1/H}(Z^H; \pi'_n) \) converges almost surely to \( \gamma_1/H \). Consequently, along this subsequence we have \( \lim_{n \to \infty} v_p(Z^H; \pi'_n) = \infty \) almost surely. Since \( |\pi'_n| \) tends to zero as \( n \) increases \( v^0_p(Z^H) \) cannot exist. This also shows that \( v_p(Z^H) = \infty \) almost surely for \( p < \frac{1}{H} \). Since \( v_p(Z^H) \) is finite almost surely for all \( p > \frac{1}{H} \), then by Lemma 1.8 and Proposition 1.6 we must have \( v(Z^H) = \frac{1}{H} \) [4]. ∎

2 Semimartingale in Fractional Brownian motion

The impossibility to receive a riskless gain by trading into a market, is a basic equilibrium assumption throughout financial mathematics. This phenomenon in finance is denoted by non-existence or absence of arbitrage. For various mathematical definitions of arbitrage and possible implications see [11], [20] and [21]. The line of arguments is put forward as follows. If we suppose that there is a strategy for the investor, the investor will choose the strategy with a riskless gain. The investor would like to buy this strategy. Depending on the market situation and by the law of supply and demand the price of this strategy would increase, immediately, showing that the market prices have not been in equilibrium. And that is why the absence of arbitrage has become a frequent minimum requirement for pricing models. In mathematics, the first fundamental theorem of asset pricing, links the no-arbitrage property to the martingale property of the discounted stock price process [11].
and references therein. The assumption of a uniform $\sigma$-field for every participant in the market at any instant of time models is a situation in which everybody has the same information, which is unrealistic. Moreover, time series analysis of stock prices exhibit intrinsic properties, so called stylized facts, see \cite{25}, which demand for more advanced mathematical models for financial markets.

Simple models of a financial market with heterogeneous interacting agents are dealing with this and are also capable of reproducing volatility clustering and long memory in time series for returns of financial prices, see e.g. the article of Alfarano et al., Cont or Kirman in \cite{25}. Finally "bubbles" and "herding" behaviour in financial markets are regarded to be incompatible with the efficient market hypothesis and the idea of rational expectation \cite{25}.

Most often the Brownian geometric motion model for the movement of the share prices is used in the theory of mathematical finance. This is incorrect empirically in a number of ways as we shall demonstrate in the last chapter with statistical tools. Many alternatives like Fractional Brownian motion have been used to account for empirically observed deficiencies. Oksendal \cite{21} (see also references therein) have developed a stochastic integral with respect to Fractional Brownian motion and a geometric Fractional Brownian motion model not admitting arbitrage, see \cite{21} for precise definition. More often, Fractional Brownian motion is used to model volatility e.g. in ARCH models and their generalization. We are going to show that Fractional Brownian motion is not a semimartingale which results in absence of an equivalent martingale measure. By the first fundamental theorem of asset pricing this means that there must be arbitrage \cite{20} in the sense used by Rogers \cite{11}.

Let $(B_t)_{t \in \mathbb{R}}$ be a standard Brownian motion with $B_0 = 0$, where the paths of $B$ are continuous and the increments of $B$ over disjoint time intervals are independent zero-mean Gaussian random variables with variance equal to the length of the interval. Consider a basic probability space $(\Omega, \mathcal{F}, P)$ with a filtration $\{\mathcal{F}_t\}$ to which $(B_t)_{t \in \Omega}$ is adapted, then

- $E[|B_t|] = E[1 \cdot |B_t|] \leq 1 \cdot E[B_t^2]^{1/2} = t < \infty$.
- For $0 \leq s \leq t$ yields
\[
E[B_t | \mathcal{F}_s] = E[B_t - B_s | \mathcal{F}_s] + E[B_s | \mathcal{F}_s] = E[B_t - B_s] + B_s = B_s,
\]
where we have used Cauchy Schwarz inequality for the first result and $B_t - B_s \perp \mathcal{F}_s$ together with $B_s \in \mathcal{F}_s$ for the second.

**Definition 2.1.** A continuous semimartingales is a stochastic processes $X = (X_t)_{t \geq 0}$ representable as sums
\[
X_t = X_0 + M_t + A_t,
\]
where $A = (A_t, \mathcal{F}_t)_{t \geq 0}$ is a process of bounded variation and $M = (M_t, \mathcal{F}_t)_{t \geq 0}$ is a local martingale both defined on some filtered probability space
\[
(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)
\]
see \cite{7}.

There are various ways of representing Fractional Brownian motion as a stochastic integral with respect to Brownian motion using different deterministic integrands. Following \cite{14} Fractional Brownian motion $(Z_t)_{t \in \mathbb{R}}$ with self-similarity parameter $H \in (0, 1)$
is e.g. given by
\[ Z_t = k \left[ \int_{-\infty}^t (t-s)^{H-\frac{1}{2}} dB_s - \int_{-\infty}^0 (s)^{H-\frac{1}{2}} dB_s \right], \tag{9} \]
where
\[ k^{-2} = \frac{1}{2H} + \int_0^\infty ((1+v)^{H-\frac{1}{2}} - v^{H-\frac{1}{2}})^2 dv. \]
The process \( Z \) is a zero-mean Gaussian process and the constant \( k \) stands for normalizing the covariance structure. Actually equation (9) cannot be defined in this way, see [11] and the correct definition is
\[ Z_t \equiv k \int_{-\infty}^\infty \left\{ ((t-s)^{+})^{H-\frac{1}{2}} - (s^-)^{H-\frac{1}{2}} \right\} dB_s, \]
where the plus and minus, respectively, are standing for positive and negative evolution. An alternative way of representing Fractional Brownian motion is given in (14).

The case \( H = \frac{1}{2} \) corresponds, as mentioned before, to the familiar situation of Brownian motion.

From (10) we see that the process of increments of \( Z \) is stationary and \( Z_{t+\delta} - Z_t \sim \delta^H \), which suggest that
\[ \sum_{j=1}^{2^n} |Z(j2^{-n}) - Z((j-1)2^{-n})|^p \sim (2^n)^{1-pH}. \tag{11} \]

If we let \( n \to \infty \), we expect that the \( p \)-variation of \( Z \) will be infinite if \( p < \frac{1}{H} \), and it will be zero if \( p > \frac{1}{H} \). This is only consistent with semimartingale behavior if \( H = \frac{1}{2} \). Where the use of the symbol \( \sim \) is in the conventional sense, that is, \( a_n \sim b_n \) means \( \frac{a_n}{b_n} \to a.e. \) as \( n \to \infty \).

Now, constructing an arbitrage is given by [11]. For \( t > 0 \) we have
\[ E(Z_t | \mathcal{F}_0) = \int_{-\infty}^0 \left\{ (t-s)^{H-\frac{1}{2}} - (s)^{H-\frac{1}{2}} \right\} dB_s. \tag{12} \]
where \( \mathcal{F}_t = \sigma(\{B_u : u \leq t\}) \), see [11]. Now if we set \( E(Z_t | \mathcal{F}_0) = E(Z_t | \mathcal{F}_0) \), with \( \mathcal{F}_t = \sigma(\{Z_u : u \leq t\}) \), we conclude from (12) that \( (Z_u)_{u \leq t} \) gives us information about the future behaviour of \( Z \), except in the case of \( H = \frac{1}{2} \). If \( E(Z_t | \mathcal{F}_0) > 0 \), we could make a positive investment in the asset. But if \( E(Z_t | \mathcal{F}_0) < 0 \) we would short the asset making a profit both ways [11]. This example illustrates that subtle and mathematically refined techniques are needed when attempting to model with Fractional Brownian motion in the context of asset pricing [21].

**Proposition 2.1.** Fractional Brownian motion with Hurst-index \( H \neq \frac{1}{2} \) is not a semimartingale.

**Proof.** For the case of \( H < \frac{1}{2} \) we know by Proposition [1,10] that Fractional Brownian motion has no-quadratic variation. Any semimartingale can be decomposed into a process of limit variation i.e. with vanishing quadratic variation and a local martingale having locally finite quadratic variation according to Doob-Meyer theorem for semimartingale. Then we conclude that it cannot be a semimartingale [22].
Now, we suppose that \( H > \frac{1}{2} \) and assume that Fractional Brownian motion is a semi-martingale with decomposition \( Z = A + M \), where \( M \) is a local martingale and \( A \) is a process of finite variation starting from 0. By Proposition 1.10 we can establish that \( Z \) has zero quadratic variation. Since \( Z \) is continuous then by the semimartingale decomposition theorem we obtain that \( M \) is also continuous. But a continuous martingale with zero quadratic variation is a constant \( M_0 \). Then \( Z = A + M_0 \) and \( Z \) must be of bounded variation. This is a contradiction since \( \nu_1(Z) \geq \nu_p(Z) = \infty \) for all \( p \leq \frac{1}{H} \).

The result in (11) implies that for \( H \neq \frac{1}{2} \) the process \( Z \) is not a semimartingale and therefore there can be no equivalent probability under which \( Z \) becomes a martingale. And the existence of an equivalent martingale measure is equivalent to NFL VR (no free lunch with vanishing risk) condition [20]. This condition is more restrictive than the condition of no arbitrage. This implies that there is arbitrage. In [11] Rogers specifies that there exists an arbitrage possibility if there is some trading strategy whose gains process \( (\xi_t)_{0 \leq t < 1} \) satisfies

- \( \xi_0 = 0 \leq \xi_1 \)
- \( \xi_t \geq -1 \) for all \( 0 \leq t < 1 \)
- \( P(\xi_1 > 0) > 0 \).

For a different definition see e.g. [21].

3 Integration of Fractional Brownian motion

We have seen that \( Z \) is not a semimartingale. That means that the theory of Gaussian process ought to be applied rather than the usual martingale approach for constructing the integral. Actually we shall consider deterministic integrands only.

Let \( f \) denote the integral operator

\[
\Gamma f(t) = H(2H - 1) \int_0^\infty f(s) |t - s|^{2H-2} ds.
\]

We define an inner product by the following

\[
< f, g > = \int_0^\infty \int_0^\infty f(s)g(t)|t - s|^{2H-2}dsdt,
\]

where \( < \cdot, \cdot > \) denotes the usual inner product of \( L^2([0, \infty)) \). Let \( L^2_\Gamma \) be the space of equivalence classes of measurable functions \( f \) such that \( < f, f >_\Gamma < \infty \). Comparing with (9), we are able to extend \( Z_t \mapsto 1_{[0,t]} \) to an isometry between the Gaussian space generated by the random variables \( Z_t, t \geq 0 \), (as the smallest closed linear subspace of \( L^2([0, \infty)) \)) and the function space \( L^2_\Gamma \) [14]. Then the integral \( \int_0^\infty f(t)dz_t \), which we construct below, can be defined as the image of \( f \) in this isometry. This corresponds to the first step in the construction of an Itô integral with the additional operator \( \Gamma \). As in the well known Itô case the construction is not pathwise.

For the case of \( H < \frac{1}{2} \), the previous integral (13) in the definition of \( \Gamma \) diverges. The operator will be defined as

\[
\Gamma f(t) = f(0-) = 0. \text{ Here we identify again } Z_t \mapsto 1_{[0,t]} \text{ up to an isometry} [14].
\]
3.1 The construction of the integral

We give an alternative construction of a stochastic integral with respect to Fractional Brownian motion which can be extended to random integrands [3]. The Gauss hypergeometric function $F(a, b, c, z)$ is defined for any $a, b, c, |z| < 1$ and for any $c \neq 0, -1, \ldots$ by

\[ F(a, b, c, z) = \sum_{k=0}^{+\infty} \frac{(a)_k(b)_k}{(c)_kK!} z^k, \]

where $(a)_0 = 1$ and $(a)_k = a(a+1)\ldots(a+k-1)$ is the Pochhammer symbol [24]. We use the following representation of Fractional Brownian motion as in [3]

\[ Z^H(t) = \int_0^t K^H(t, s)dB(s), \quad (14) \]

where

\[ K^H(t, s) = \frac{(t-s)^{H-\frac{1}{2}}}{\Gamma(H+\frac{1}{2})} F \left( H - \frac{1}{2}, \frac{1}{2}, H + \frac{1}{2}, \frac{1}{s} \right), \quad s < t \quad (15) \]

where $F$ denotes Gaussian hypergeometric function and $(B(t); t \geq 0)$ is standard Brownian motion. We observe that $K^H(t, s) = I^H_{[0,t]}(s)$ where $I^H_t$ is the integral operator,

\[ I^H_t f(s) = K^H(t, s)f(s) + \int_s^t (f(u) - f(s))\partial_1 K^H(u, s)du. \quad (16) \]

Now we define

\[ \int_0^t f(s)dZ^H(s) = \int_0^1 I^H_t f(u)dB_u \quad (17) \]

for suitable deterministic functions.

We introduce an approximation to $Z^H$ by processes of the type

\[ Z_K(t) = \int_0^t K(t, s)dB(s), \quad t \in [0, T] \]

with kernels $K$ which prefers to be smooth enough to ensure that $Z_K$ is a semi martingale, according to Proposition 2.5 in [3].

3.2 Integration of the deterministic function

We recall (16). For a kernel $K$ we associate the integral operator for a measurable $a$ on $(0, t)$

\[ I_t a(s) = K(t, s)a(s) + \int_s^t (a(u) - a(s))\partial_1 K(u, s)du, \quad 0 < s < t, \]

when the integral of the right-hand side makes sense for almost every $s$ in $(0, t)$. Now, when $K$ is a smooth kernel, we have

\[ I_t a(s) = K(s, s)a(s) + \int_s^t a(u)\partial_1 K(u, s)du, \quad 0 < s < t. \]

We also consider the integral operator

\[ J_t a(s) = \int_s^t a(u, s)\partial_1 K(u, s)du, \quad 0 < s < t. \]
Since \( K(t, s) = I_t 1_{[0, t]}(s) \), we have

\[
\text{Cov}(Z_K(s), Z_K(t)) = E \left[ \int_0^t K(t, u) dB_u \int_0^s K(s, v) dB_v \right]
\]

(18)

\[
= < I_t 1_{[0, t]}, I_t 1_{[0, s]} >_{L^2},
\]

(19)

where the inner product is of \( L^2(\mathbb{R}) \).

We define \( L^2_K(0, t) = I_t^{-1}(L^2(0, t)) \). With respect to the inner product we obtain

\[
< f, g >_{L^2_K(0, t)} \overset{def}{=} < I_t f, I_t g >_{L^2}.
\]

Given \( f \in L^2_K(0, t) \), it defines

\[
\int_0^t f(s) dZ_K(s) = \int_0^t I_t f(u) dB_u.
\]

4 Statistical methods in mathematical finance

Refering to the notation in Paragraph 1.1, we call the sequence \( Y = (Y_n)_{n \geq 1} \) Fractional Gaussian noise with Hurst-index \( H \), \( 0 < H < 1 \).

The case of \( \frac{1}{2} < H < 1 \) is characteristic of a long memory or a strong affect. We can find such a phenomena in the widths of annual rings of trees, in the behavior of rivers and in the absolute values of the returns for stock prices.

In the easiest version of the Black Scholes model, prices of shares are given by

\[
S_t = e^{-rt - \frac{1}{2} \sigma^2 t + \sigma B_t},
\]

(20)

where \( r \) is the interest rate. The values of the returns are given empirically by \( h_n = \ln \frac{S_n}{S_{n-1}} \), \( n \geq 0 \), for discrete time steps.

If \( H = \frac{1}{2} \), the standard deviation \( \sqrt{D(h_1 + \cdots + h_n)} \) increases with \( \sqrt{n} \). If \( H > \frac{1}{2} \), then the growth is of order \( n^H \). In other words, the diffusion of the values of the resulting variable \( H_n = h_1 + \cdots + h_n \) is larger than for the case of \( H = \frac{1}{2} \).

On the other hand, Fractional noise \( h = (h_n) \) has a negative covariance for the case of \( 0 < H < \frac{1}{2} \), as mentioned before, which corresponds to fast changing of the values of the \( h_n \). This is also characteristic of turbulence phenomena [7] which indicates that Fractional Brownian motion with \( 0 < H < \frac{1}{2} \) can serve as a fair model of turbulence.

The concept of volatility has become a concept of growing interest in finance in recent years. To many people volatility means turbulence, but volatility can be modeled by the standard deviation of stock price changes.

One uses the standard deviation because it measures the dispersion of the percentage of change in prices or the return of the probability distribution. One may replace the non deterministic constant \( \sigma \) in (20) by a random process \( \sigma_t \). The larger the standard deviation is, the higher the probability of a large price change and the riskier the stock.

The standard deviation will tend to a value that is the population standard deviation in case of the assumptions that the returns are sampled from a normal distribution and that the variance is finite. This is why standard deviation has become known as a measure of the volatility. Volatility has become even more important measure because for option pricing formula of Black-Scholes.
4.1 Rescaled range method

We now introduce a method which is regularly used to study the Hurst index of a time serial. Convergence was proven by William Feller [17]. Let us consider a time serial of random variables of consecutive instants of time. For example the payoffs of 12 occasions constitute random variables \( h(1), h(2), \ldots, h(12) \). Assuming that \( h \) has a finite variance then we let \( m \) be the sample mean and \( c \) be the sample standard deviation. The first we do is to remove any tendency that makes \( h \) to rise or fall as a long run process. So we do the following

\[
\begin{align*}
  x(1) &= h(1) - m, \\
  x(2) &= h(2) - m, \\
  &\vdots \\
  x(12) &= h(12) - m.
\end{align*}
\]

We obtained a set of random variables \( x \) with mean zero. Now we form partial sums of these random variables, denoted by \( y(n) \)

\[
\begin{align*}
  y(1) &= x(1), \\
  y(2) &= x(1) + x(2), \\
  &\vdots \\
  y(12) &= x(1) + x(2) + \ldots + x(12).
\end{align*}
\]

We obtain a new set \( \{y(1), \ldots, y(12)\} \), where each \( y(n) \) is the sum of mean-zero random variables \( x \). We denote \( R = \max(y) - \min(y) \) to be the range. If we rescale with the standard deviation \( c \), we get the scaled range \( R/c \). Feller [17] had proven that if the series of random variables were independent and had a finite variance, then the rescaled range would be

\[
\frac{R}{c} = kn^\frac{1}{2}
\]

where \( k \) is a constant. The hydrologist, Hurst, found that the general rescaled range was given by

\[
\frac{R}{c} = kn^H.
\]

The latter equation can equally be written as \( \ln \left( \frac{R}{c} \right) = \ln k + H \ln n \). Then we can estimate the value of \( H \) by running a regression of \( \ln \left( \frac{R}{c} \right) \) against \( \ln n \). Applying this method to the serial data taken from a daily OMX, Stockholm Stock Exchange, OMX Futures 1-Pos from October 27, 1999 to Jun 11, 2007. But at first we convert the data into a series of log differences

\[
h_t = \ln(S_t) - \ln(S_{t-1}),
\]

where \( h_t \) is log return at time \( t \) and \( S_t \) is the price at time \( t \). Then we obtain that \( H = 0.57 \). An other way of estimating the payoffs is by R/S method, which is an interesting subject to study because we make so many assumptions with few facts to back us up.

4.2 The R/S method

Let \( S_t, t \geq 0 \), be a time series of stock prices. We use the same serial as above. We try now to check the R/S method for computing the Hurst-index on data generated by simulating Fractional Brownian motion [16].
We begin by dividing the serial in $K$ non-intersecting blocks that all contain $M$ elements, where $M$ is the greatest integer that is smaller than $\frac{N}{K}$. The rescaled adjusted range $R(t_i, r) / S(t_i, r)$ is then computed for a number of ranges $r$. We calculate $R(t_i, r)$ by the following

$$R(t_i, r) = \max\{W(t_i, 1), \ldots, W(t_i, r)\} - \min\{W(t_i, 1), \ldots, W(t_i, r)\},$$

where

$$W(t_i, k) = \sum_{j=0}^{k-1} X_{t_i+j} - k \left( \frac{1}{r} \sum_{j=0}^{r-1} X_{t_i+j} \right) = \sum_{j=0}^{k-1} \left( X_{t_i+j} - k \bar{X}_{r,i} \right),$$

for $1 \leq k \leq r \leq M$, and $t_i = M(i - 1)$ are the starting points of the blocks, $i = 1, \ldots, K$. The quantity

$$\bar{X}_{r,i} = \frac{1}{r} \sum_{j=0}^{r-1} X_{t_i+j}$$

is the empirical mean of the sample $(X_{t_i}, \ldots, X_{t_i+r-1})$. Therefore $W(t_i, k)$ is the deviation of $\sum_{j=0}^{k-1} X_{t_i+j}$ from the empirical mean value $k \bar{X}_{r,i}$. Moreover, the range $R$ only can be computed if $t_i + r \leq N$. The sample variance $s^2(t_i, r)$ of $X_{t_i}, \ldots, X_{t_i+r-1}$ is given by

$$s^2(t_i, r) = \frac{1}{r} \sum_{j=0}^{r-1} X_{t_i+j}^2 - \left( \frac{1}{r} \sum_{j=0}^{r-1} X_{t_i+j} \right)^2 = \frac{1}{r} \sum_{j=0}^{r-1} \left( X_{t_i} - \bar{X}_{r,i} \right)^2.$$

We obtain a number of R/S samples for each value of $r$. When $r$ is small we can compute all the samples since $t_i + r \leq N$ for all $i = 1, \ldots, K$. The number of samples decreases for increasing value of $r$. And the number of R/S samples approaches 1 as $r$ approaches $N$.

For Fractional Gaussian noise, the R/S statistic is proportional to $r^H$ as $r \to \infty$. Setting the averages to zero, $W(t_i, k)$ is Fractional Brownian motion sample of size $r$. The range $R(t_i, r)$ is the difference between the maximum and the minimum of the sample. Dividing
by the square root of the variance, we obtain that the R/S statistic is proportional to $r^H$ as well. We set up a linear regression model in order to determine the $H$-value. In order to be able to apply linear regression we transform the $R/S$ by a logarithm. The deterministic component of $\ln r$ and the corresponding stochastic component is given by $\ln R/S(\cdot, r)$. For each value of $r$ we find the $k$ realizations of the random variable $R_i/S_i$, $1 \leq i \leq k$ which regularly happen for the regression procedure.

If we simulate Fractional Brownian motion with $H = 0.57$ we obtain the values as in figure 4.2.

Now, if we use the original data from a time serial of stock prices $S_t$, Stockholm Stock Exchange, OMX Futures 1-Pos and converting the data into a series of log differences

$$h_t = \ln(S_t) - \ln(S_{t-1}),$$

where $h_t$ is log return at time $t$ and $S_t$ is the price at time $t$ we obtain a $H$ value of 0.61, as figure 4.3 shows. This algorithm is described in details in [15].
Conclusion

For Hurst-index different from $1/2$ the $p$-variation of the process contradicts Fractional Brownian motion being a semimartingale. This in turn might lead to the conclusion that Fractional Brownian motion other than Brownian motion is inadequate for modeling in financial markets since the first fundamental theorem of asset pricing states that the existence of an equivalent martingale measure corresponds to non existence of riskless gains usually called arbitrage free [21]. This equilibrium assumption, however, is accepted in the field of financial mathematics. The R/S analysis reveals that there do exist shares the daily pay offs of which exhibit a fractional structure different from that of Brownian motion. Moreover it is well known that longer term pay offs of all shares have this property. Fractional Brownian motion is also successfully used in other fields of mathematical finance, for example in studying random volatilities.

There are classical generalizations of stochastic integrals of Itô type which are based on semimartingales. The construction of a stochastic integral with respect to Fractional Brownian motion leads to different procedures. We present a construction of a stochastic integral with respect to Fractional Brownian motion which makes it possible to integrate deterministic functions. This construction shown is the one most similar to constructing an Itô integral.

Figure 4.3: The regression line is $-0.26 + 0.61z$ with Hurst-index $H = 0.61$. 

![Regression Line](image-url)
5 Appendix

To find the Hurst index using $R/S$ method we use the following algorithm

FindMaxExponentOfTwo := proc (n::integer) local t; t := 2; while $2^t \leq n$ do t := t+1 end do; return t-1 end proc;

RSHurstExponent := proc (A::array)
local T, startSize, M, maxExpTwo, expo,
blockSize, blockNo, avg, blockStart, blockEnd, i, j, k, blockMin, blockMax, R,
S, X, RS, t, l, Points, XV, YV, regLine, plot1, plot2; startSize := 8;
Points[ ] := [ ]; XV[ ] := [ ]; YV[ ] := [ ]; M := op(2,op(2,eval(A)));
maxExpTwo := FindMaxExponentOfTwo(M); T := $2^\text{maxExpTwo}$;
if M < startSize then
print("Too few data: at least ",startSize," required");
return
end if;
if T <> M then
M := T; print("Data size is not of the form $2^k$. Taking only the first",M)
end if;
RS := array(3 .. maxExpTwo-1);
for expo from 3 to maxExpTwo-1 do
blockSize := $2^\text{expo}$; blockNo := M/blockSize;
print("Handeling ",blockNo,
" blocks of size ",blockSize); avg := array(1 .. blockNo);
R := array(1 .. blockNo);
S := array(1 .. blockNo);
for i to blockNo do avg[i] := 0.;
blockStart := (i-1)*blockSize+1; blockEnd := blockStart+blockSize-1; for j from blockStart to blockEnd do
avg[i] := avg[i]/ blockSize;
X := array(blockStart .. blockEnd);
X[blockStart] := A[blockStart]-avg[i]; for j from blockStart+1 to blockEnd do
end do;
blockMin := X[blockStart]; blockMax := X[blockStart]; for l from blockStart+1 to blockEnd do blockMin := min(blockMin,X[l]);
blockMax := max(
blockMax,X[l])
end do; R[i] := blockMax-blockMin; S[i] := 0.;
for j from blockStart to blockEnd do
S[i] := S[i]+(A[j]-avg[i])^2
end do;
S[i] := sqrt(S[i]/blockSize)
end do;
RS[expo] := 0.;

\footnote{By using Maple}
for t to blockNo do
RS[expo] := RS[expo]/blockNo;
Points := [op(Points), [log( blockSize), log(RS[expo])]]; XV := [op(XV), log(blockSize), log(RS[expo])];
XV := [op(XV), log(RS[expo])];

plot1 := pointplot(Points, scaling = constrained);
print("Making point plot", pointplot(Points, scaling = constrained));
regLine := Statistics[LinearFit]([1, z], XV, YV, z); plot2 := plot(regLine, z = 3 .. log(2^((maxExpTwo-1))));
print(display([plot1, plot2]));
print("The regression line is ", regLine); print("The Hurst exponent is ", op(1, op(2, regLine))); return Points end proc;

We can estimate Hurst index with help of the method described in [16] according to the following:

covariance := proc(i, H); autocovariance function of fractional Gaussian noise
if (i = 0) then
return (1);
else return ( ((i-1)^(2*H)-2*i^(2*H)+(i+1)^(2*H))/2 );
fi; end;
simFBM := proc(n, H, L, accu)
generates a sample of size 2^n of fractional Gaussian noise if accu=0
# fractional Brownian motion if accu<>0
# The Hurst index is H, the values are in the # range [0;L]
local i, j, m, v, scaling, phi, psi, result, cov, stdnorm, rr;
m := 2^n;
phi := array(0..m-1);
psi := array(0..m-1);
cov := array(0..m-1);
result := array(0..m-1);
rr := array(1..m);
stdnorm := [stats[random, normald[0.0, 1.0]](m)];
# initialization
result[0] := stdnorm[1];
v := 1.0;
phi[0] := 0.0;
for i from 0 to m-1 do
cov[i] := covariance(i, H);
od;
# producing the values
for i from 1 to m-1 do
phi[i-1] := cov[i];
for j from 0 to i-2 do
psi[j] := phi[j];
phi[i-1] := phi[i-1] - psi[j]*cov[i-j-1];
end do;
if (v > 0) then
    phi[i-1] := phi[i-1]/v;
else
    v := 0.9;
fi;
for j from 0 to i-2 do
    phi[j] := psi[j] - phi[i-1]*psi[i-j-2];
od;
v := v * (1 - phi[i-1]*phi[i-1]);
result[i] := 0.0;
for j from 0 to i-1 do
    result[i] := result[i] + phi[j]*result[i-j-1];
od;
result[i] := result[i] + sqrt(v)*stdnorm[i+1];
od;
# scale to range [0,L]
scaling := (L/m)^H;
for i from 0 to m-1 do
    result[i] := scaling*result[i];
    if ((accu = 1) and (i>0))
        then
            result[i] := result[i] + result[i-1];
            fi;
    od;
for i from 1 to m do
    rr[i] := result[i-1];
od;
return(rr);
end;
The algorithm of [17] uses overlapping intervals.

SimpleHurstExponent := proc(A::array)
local mean, k, n, x, y, sumSqr, scale, i, hurst, maxY, minY, range;
n := op(2,op(2,eval(A)));
k comes from an analysis of Feller, see http://www.aci.net/kalliste/chaos7.htm
k := evalf(sqrt(Pi/2));
x := array(1..n);
y := array(1..n);
mean := 0.0;
for i from 1 to n do
    mean := mean + A[i];
    od;
mean := mean / n;
sumSqr := 0.0;
for i from 1 to n do
    x[i] := A[i] - mean;
    sumSqr := sumSqr + x[i]*x[i];
    od;
Here we present a calculation of (5). Choosing different values of the lists length and the enumeration of the lists we obtain the corresponding Hurst index.
In order to visualize the path of Fractional Brownian motion we use the algorithm of [19].

By using Mathematica

```mathematica
<< fbm.m
Needs["MathProg`fBm""]
Needs["Graphics`ParametricPlot3D"]
SetOptions[{SurfaceGraphics, Plot3D}, MeshStyle -> Thickness[0], Mesh -> False];
SetOptions[{ContourGraphics, ContourPlot}, ContourStyle -> Thickness[0]; SetOptions[ListPlot, PlotStyle -> Thickness[0]];
n1 = 512;
Do[h9 = ListPlot[fBmRA[1, n1, h], PlotJoined -> True, PlotRange -> All, AspectRatio -> 0.3], {h, 0.9, 0.9, -0.2}];
Do[h5 = ListPlot[fBmRA[1, n1, h], PlotJoined -> True, PlotRange -> All, AspectRatio -> 0.3], {h, 0.9, 0.5, -0.2}];
Do[h1 = ListPlot[fBmRA[1, n1, h], PlotJoined -> True, PlotRange -> All, AspectRatio -> 0.3], {h, 0.9, 0.1, -0.2}];
```
References


