Measurements of the branching fractions of $\eta_c \to K^+K^0\pi^0$, $K_S^0K^0\pi^0$, $2(\pi^+\pi^-\pi^0)$, and $p\bar{p}$
I. INTRODUCTION

Many new charmonium or charmoniumlike states have been discovered recently [1], which broaden our horizon on understanding the charmonium family. These states have led to a revived interest in improving the quark-model picture of hadrons. However, the knowledge of the lowest lying charmonium state, $\eta_c$, is relatively poor compared to other charmonium states. The reason is that most of the measurements involving $\eta_c$ were performed using the magnetic dipole (M1) transitions from $J/\psi$ or hindered $M1$ transitions from $\psi(3686)$. In these decays, the interference between $\eta_c$ and non-$\eta_c$ amplitudes affects the $\eta_c$ line shape [2]. The branching fraction (BF) of $\eta_c$ decays and the $M1$ transition rate are entangled. The insufficient understanding of the $\eta_c$ properties has so far prevented precise studies of $\eta_c$ decays themselves or of decays involving the $\eta_c$. For example, in 2002, the Belle Collaboration released the measurements on the total cross section of the exclusive production of $J/\psi + \eta_c$ via the $e^+e^-$ annihilation at the center-of-mass energy $\sqrt{s} = 10.58$ GeV [3] with the result of $\sigma(e^+e^- \rightarrow J/\psi + \eta_c) \times BF(\eta_c \rightarrow \geq 4 \text{ charged}) = 33^{+7}_{-6} \pm 9 \text{ fb}$. These measurements were improved as $\sigma(e^+e^- \rightarrow J/\psi + \eta_c) \times BF(\eta_c \rightarrow \geq 2 \text{ charged}) = 25.6 \pm 2.8 \pm 3.4 \text{ fb}$ [4]. In 2005, the BABAR Collaboration independently measured the total cross section as $17.6 \pm 2.8^{+1.5}_{-2.1}$ fb [5]. As the number of charged tracks is required in these measurements, the results will be improved if the charged tracks multiplicity is fully studied.

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Recently, the electric dipole (E1) transition \( h_c \rightarrow \eta_c \) was found to be a perfect process to measure both \( \eta_c \) resonant parameters and its decay BFs [6]. In addition, the \( h_c \) production proceeds via \( \psi(3686) \rightarrow \pi^0 h_c \), where the interference effect between \( \eta_c \) and non-\( \eta_c \) is much less than that in \( J/\psi, \psi(3686) \) radiative transition. One can draw such a conclusion according to the following calculation. The E1 transition rate, \( BF(h_c \rightarrow \eta_c) = 50\% \), is about 2 orders of magnitude larger than that of the M1 transition \( BF(\psi(3686) \rightarrow \eta_c) = 0.3\% \) [7]. On the other hand, the background that can interfere with the signal comes from charmonium radiative decays, e.g., \( h_c, \psi(3686) \rightarrow \gamma + \text{hadrons} \). If we assume the radiative decay rates of \( h_c \) and \( \psi(3686) \) to be at the same level, therefore, this kind of background in the process \( h_c \rightarrow \gamma \eta_c \) should be 1 to 2 orders of magnitude less than in \( \psi(3686) \rightarrow \gamma \eta_c \).

BESIII has collected sizable data samples between 4.009 and 4.600 GeV (called “XYZ data” hereafter) since 2013 to study the XYZ states [8]. A large production rate of \( e^+ e^- \rightarrow \pi^+ \pi^- h_c \) has been found [9]. The total number of \( h_c \) events in all these data samples combined is comparable to that from \( \psi(3686) \rightarrow \pi^0 h_c \) decays in BESIII data, according to the measured cross section and the corresponding integrated luminosity at each energy point. The \( h_c \) is tagged by the recoil mass (RM) of \( \pi^+ \pi^- \) in XYZ data, while it is tagged by the recoil mass of \( \pi^0 \) in \( \psi(3686) \) data. Generally, the two-charged-pion mode has lower background and higher detection efficiency than the neutral pion mode.

In this paper, we report a measurement of the BFs of four \( \eta_c \) exclusive decays via the process \( e^+ e^- \rightarrow \pi^+ \pi^- h_c \), \( h_c \rightarrow \gamma \eta_c \). These exclusive decays are \( \eta_c \rightarrow K^+ K^- \pi^0 \), \( K_S^0 K^+ \pi^- \), \( 2(\pi^+ \pi^- \pi^0) \), and \( pp \), respectively.

Apart from the BF measurement mentioned above, we also measure the charged tracks multiplicities in inclusive \( \eta_c \) decays by using an unfolding method [10].

II. METHODOLOGY

The BFs of \( \eta_c \) exclusive decays are obtained by a simultaneous fit to the RM spectrum of \( \pi^+ \pi^- \gamma \) for both inclusive and exclusive modes. The BFs are common parameters independent of the center of mass energy. The numbers of the \( \eta_c \) signal events of the exclusive and inclusive decay modes can be calculated by the following formulas,

\[
N_{\text{exclusive}}^i = \mathcal{L}^i \times \sigma(e^+ e^- \rightarrow \pi^+ \pi^- h_c) \times BF(h_c \rightarrow \gamma \eta_c) \times BF(\eta_c \rightarrow X) \times BF(X \rightarrow Y) \times \epsilon_{\text{exclusive}}^i, \tag{1}
\]

and

\[
N_{\text{inclusive}}^i = \mathcal{L}^i \times \sigma(e^+ e^- \rightarrow \pi^+ \pi^- h_c) \times BF(h_c \rightarrow \gamma \eta_c) \times \epsilon_{\text{inclusive}}^i, \tag{2}
\]

where the subscript \( i \) denotes the different center-of-mass energy points. \( \mathcal{L} \) and \( \sigma \) denote the luminosity and cross section, respectively. \( X \) denotes a certain \( \eta_c \) exclusive decay mode, \( Y \) denotes the possible \( \pi^0 \) or \( K_S^0 \) final state from \( X \) decay, \( \epsilon \) denotes the detection efficiency determined by Monte Carlo (MC) simulations.

By comparing Eq. (1) and Eq. (2), \( BF(\eta_c \rightarrow X) \) can be extracted as

\[
BF(\eta_c \rightarrow X) = \frac{N_{\text{exclusive}}^i / (BF(X \rightarrow Y) \times \epsilon_{\text{exclusive}}^i)}{N_{\text{inclusive}}^i / \epsilon_{\text{inclusive}}^i}. \tag{3}
\]

In the simultaneous fit, the total number of free parameters is less than in the fits taken individually, due to common parameters such as the \( \eta_c \) mass and width, etc. In addition, some parameters, for example, \( \sigma(e^+ e^- \rightarrow \pi^+ \pi^- h_c) \), \( \mathcal{L} \), are not necessary in the measurement according to Eq. (3), resulting in reduced statistical uncertainties. In addition, systematic uncertainties from the same sources, e.g., the tracking efficiency of two pions from \( e^+ e^- \rightarrow \pi^+ \pi^- h_c \), can be canceled.

III. DETECTOR AND DATA SAMPLES

The BESIII detector is a magnetic spectrometer [11] located at the Beijing Electron Positron Collider (BEPCCII) [12]. The cylindrical core of the BESIII detector consists of a helium-based multilayer drift chamber (MDC), a plastic scintillator time-of-flight system (TOF), and a CsI(Tl) electromagnetic calorimeter (EMC), which are all enclosed in a superconducting solenoidal magnet providing a 1.0 T magnetic field. The solenoid is supported by an octagonal flux-return yoke with resistive plate counter muon identifier modules interleaved with steel. The acceptance of charged particles and photons is 93% over a 4\( \pi \) solid angle. The charged-particle momentum resolution at 1 GeV/c is 0.5%, and the specific energy loss \((dE/dx)\) resolution is 6% for the electrons from Bhabha scattering. The EMC measures photon energies with a resolution of 2.5% (5%) at 1 GeV in the barrel (end cap) region. The time resolution of the TOF barrel part is 68 ps, while that of the end cap part is 110 ps.

The data samples collected at four center-of-mass energies, i.e., \( \sqrt{s} = 4.23, 4.26, 4.36, \) and 4.42 GeV [8], are used for our studies. Simulated samples produced with the GEANT4-based [13] MC package which includes the
The geometric description of the BESIII detector and the detector response, are used to determine the detection efficiency and to estimate the backgrounds. The simulation includes the beam energy spread and initial state radiation (ISR) in the $e^+e^-$ annihilations modeled with the generator KKMC [14].

The inclusive MC samples have the equivalent luminosities the same as the data samples. They consist of the production of open charm processes, the ISR production of vector charmonium(like) states, and the continuum processes incorporated in KKMC [14]. The known decay modes ($\sim$50%) are modeled with EVTGEN [15] using branching fractions taken from PDG [7], and the remaining unknown decays ($\sim$50%) from the charmonium states with LUNDCHARM [16]. The final state radiations (FSR) from charged final state particles are incorporated with the PHOTOS package [17].

Signal MC samples with 200 000 events each are generated for each $\eta_c$ decay mode (inclusive and exclusive decays) at each center-of-mass energy. ISR is simulated using KKMC with a maximum energy for the ISR photon corresponding to the $\pi^+\pi^-h_c$ mass threshold. The $E1$ transition $h_c \to \gamma\eta_c$ is generated with an angular distribution of $1 + \cos^2 \theta$, where $\theta$ is the angle of the $E1$ photon with respect to the $h_c$ helicity direction in the $h_c$ rest frame. The inclusive decays of $\eta_c$ are produced similarly to the inclusive MC samples.

**IV. EVENT SELECTIONS**

In this analysis, the $\eta_c$ signal is tagged with $RM(\pi^+\pi^-\gamma)$ by requiring $RM(\pi^+\pi^-)$ in $h_c$ signal region. For the inclusive mode, at least two charged tracks and one photon is required. For the exclusive modes, the requirements on charged tracks and photon candidates depend on their respective final state.

Charged tracks at BESIII are reconstructed from MDC hits within a polar-angle ($\theta$) acceptance range of $|\cos \theta| < 0.93$. We require that these tracks pass within 10 cm of the interaction point in the beam direction and within 1 cm in the plane perpendicular to the beam. Tracks used in reconstructing $K_S^0$ decays are exempted from these requirements.

A vertex fit constrains charged tracks to a common production vertex, which is updated on a run-by-run basis. For each charged track, TOF and $dE/dx$ information is combined to compute particle identification (PID) confidence levels for the pion, kaon, and proton hypotheses.

Electromagnetic showers are reconstructed by clustering EMC crystal energies. Efficiency and energy resolution are improved by including energy deposits in nearby TOF counters. A photon candidate is defined as an isolated shower with an energy deposit of at least 25 MeV in the barrel region ($|\cos \theta| < 0.8$), or of at least 50 MeV in the end cap region ($0.86 < |\cos \theta| < 0.92$). Showers in the transition region between the barrel and the end cap are not well measured and are rejected. An additional requirement on the EMC hit timing suppresses electronic noise and energy deposits unrelated to the event.

A candidate $\pi^0$ is reconstructed from pairs of photons with an invariant mass in the range $|M_{\gamma\gamma} - m_{\pi^0}| < 15$ MeV/$c^2$ [7]. A one-constraint (1C) kinematic fit is performed to improve the energy resolution, with the $M_{\gamma\gamma}$ constrained to the known $\pi^0$ mass.

We reconstruct $K_S^0 \to \pi^+\pi^-$ candidates using pairs of oppositely charged tracks with an invariant mass in the range $|M_{\pi^+\pi^-} - m_{K_S^0}| < 20$ MeV/$c^2$, where $m_{K_S^0}$ is the known $K_S^0$ mass [7]. To reject random $\pi^+\pi^-$ combinations, a secondary-vertex fitting algorithm is employed to impose the kinematic constraint between the production and decay vertices [18]. Accepted $K_S^0$ candidates are required to have a decay length of at least twice the vertex resolution. If there is more than one $\pi^+\pi^-$ combinations in an events, the one with the smallest $\chi^2$ of the secondary vertex fit is retained.

In selecting the candidates of the $\eta_c$ inclusive decay, all charged tracks are assumed to be pions, and events with at least one combination satisfying $RM(\pi^+\pi^-) \in [3.46, 3.59]$ GeV/$c^2$ and $RM(\pi^+\pi^-\gamma) \in [2.52, 3.4]$ GeV/$c^2$ are kept.

**FIG. 1.** Distribution of $RM(\pi^+\pi^-)$ of the $\eta_c$ inclusive decay from signal MC simulation (a) and data (b) summed over all the four center-of-mass energies. The $h_c$ signal and sideband regions are marked by the solid and dashed arrows, respectively.
TABLE I. Requirements of the number of photons, charged tracks, \( \pi^0 \), and \( K_L^0 \) candidates in exclusive \( \eta_c \) decay modes, denoted as \( N_{\text{charge}}, N_{\gamma}, N_{\pi^0}, \) and \( N_{K_L^0} \), respectively.

<table>
<thead>
<tr>
<th>Decay mode</th>
<th>( N_{\text{charge}} )</th>
<th>( N_{\gamma} )</th>
<th>Other requirements</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \eta_c \rightarrow K^+K^-\pi^0 )</td>
<td>( =2 )</td>
<td>( \geq 3 )</td>
<td>( N_{\pi^0} \geq 1 )</td>
</tr>
<tr>
<td>( \eta_c \rightarrow K_L^0K^+\pi^- )</td>
<td>( =4 )</td>
<td>( \geq 1 )</td>
<td>( N_{K_L^0} = 1 )</td>
</tr>
<tr>
<td>( \eta_c \rightarrow 2(\pi^+\pi^-\pi^0) )</td>
<td>( =4 )</td>
<td>( \geq 5 )</td>
<td>( N_{\pi^0} \geq 2 )</td>
</tr>
<tr>
<td>( \eta_c \rightarrow p\bar{p} )</td>
<td>( =2 )</td>
<td>( \geq 1 )</td>
<td>...</td>
</tr>
</tbody>
</table>

TABLE II. The requirements of \( \chi^2_{4C} \) for the exclusive decays of \( \eta_c \).

<table>
<thead>
<tr>
<th>( \sqrt{s} ) (GeV)</th>
<th>( K^0_LK^+\pi^- )</th>
<th>( K^+K^-\pi^0 )</th>
<th>( 2(\pi^+\pi^-\pi^0) )</th>
<th>( p\bar{p} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.23 ( \pm ) 0.02</td>
<td>45</td>
<td>25</td>
<td>35</td>
<td>40</td>
</tr>
<tr>
<td>4.26 ( \pm ) 0.02</td>
<td>45</td>
<td>25</td>
<td>30</td>
<td>40</td>
</tr>
<tr>
<td>4.36 ( \pm ) 0.02</td>
<td>45</td>
<td>25</td>
<td>25</td>
<td>40</td>
</tr>
<tr>
<td>4.42 ( \pm ) 0.02</td>
<td>50</td>
<td>20</td>
<td>35</td>
<td>40</td>
</tr>
</tbody>
</table>

for further analysis. The region satisfying \( RM(\pi^+\pi^-) \in [3.515, 3.535] \) GeV/c\(^2\) is taken as the \( h_c \) signal region, while the regions satisfying \( RM(\pi^+\pi^-) \in [3.345, 3.505] \) GeV/c\(^2\) or \( RM(\pi^+\pi^-) \in [3.545, 3.555] \) GeV/c\(^2\) are taken as the \( h_c \) sidebands region. Figure 1 shows the distribution of \( RM(\pi^+\pi^-) \) for all \( \pi^+\pi^- \) combinations from the inclusive decay mode in signal MC simulations and data (summed over four-center-of-mass energies), respectively.

For the selection of exclusive \( \eta_c \) decays, the requirements on the number of photons and charged tracks are listed in Table I. A four-constraint (4C) kinematic fit imposing overall energy-momentum conservation is performed. To determine the species of final state particles and to select the best combination when additional photons (or \( \pi^0 \) candidates) are found in an event, the combination with the minimum value of \( \chi^2 = \chi^2_{4C} + \chi^2_{4C}(\text{PID}) + \sum_{i=1}^{N_{\text{charge}}} \chi^2_{\text{PID}} + \chi^2_{\text{Vertex}} \) is selected for further analysis, where \( \chi^2_{4C} \) is the \( \chi^2 \) from the four-momentum conservation kinematic fit and \( \chi^2_{4C}(\text{PID}) \) is the sum of the 1C (mass constraint of the two daughter photons) \( \chi^2 \) of the \( \pi^0 \) in the final state. \( \chi^2_{\text{PID}} \) is the \( \chi^2 \) from the PID of different particle hypothesis, using the energy loss in the MDC and the time measured with the TOF system, \( N_{\text{charge}} \) is the number of the charged tracks in the final states. \( \chi^2_{\text{Vertex}} \) is the \( \chi^2 \) of the vertex fit in \( K_L^0 \) reconstruction. The \( \chi^2_{4C} \) is required to be not more than 50 depending on the \( \eta_c \) decay modes, which is optimized using the figure of merit \( N_{\gamma}/\sqrt{N_{\gamma}+N_B} \), where \( N_{\gamma} \) is the number of signal events obtained from MC simulation (normalized to data luminosity), while \( N_B \) is the number of background events obtained from the sidebands of \( h_c \) in data. The requirement on \( \chi^2_{4C} \) for the different exclusive decay modes are listed in Table II. In addition, we require the same \( h_c \) mass windows on the \( RM(\pi^+\pi^-) \) spectra for both inclusive and exclusive modes.

V. NUMERICAL RESULTS OF BF(\(\eta_c \rightarrow X\))

A simultaneous unbinned maximum likelihood fit to the \( RM(\pi^+\pi^-\gamma) \) spectrum of the exclusive decays and the inclusive decay of \( \eta_c \) at the four center-of-mass energies is performed to obtain the branching fractions BF(\(\eta_c \rightarrow X\)). The fit function is parametrized as follows:

\[
F(M) = \sigma \otimes [\epsilon(M) \times [BW(M)]^2 \times E_{\gamma}^3 \times f_\gamma(E_{\gamma})] + B(M),
\]

where the signal function is described by a Breit-Wigner function, \( BW(M) \), convolved with the detection resolution, \( \sigma \). The mass and width of \( BW(M) \) are fixed to the \( \eta_c \) nominal values taken from the PDG [7]. \( M \) represents the recoil mass \( RM(\pi^+\pi^-\gamma) \). The detection resolution is described by a double Gaussian function, whose parameters are obtained from MC simulations. \( \epsilon(M) \) is the efficiency curve, obtained from a fit of the efficiencies along the \( RM(\pi^+\pi^-\gamma) \) spectrum with a polynomial function and fixed in the fit to data. Figure 2 shows the efficiencies along the \( RM(\pi^+\pi^-\gamma) \) spectrum for the inclusive \( \eta_c \) decay and the exclusive decay \( \eta_c \rightarrow K^+K^-\pi^0 \) at \( \sqrt{s} = 4.23 \) GeV.

FIG. 2. Efficiencies along the \( RM(\pi^+\pi^-\gamma) \) spectra from MC simulation at \( \sqrt{s} = 4.23 \) GeV for inclusive decay (a) and \( \eta_c \rightarrow K^+K^-\pi^0 \) (b). The curves are the fit results.
\[ E_\gamma = (m_{h_c}^2 - M^2) / 2m_{h_c} \] is the energy of the transition photon, where \( m_{h_c} \) is the \( h_c \) mass \([7]\).

\[ f_d(E_\gamma) = \frac{E_0^2}{E_0^2 + (E_\gamma - E_0)^2} \]

is the damping factor \([19]\), where \( E_0 = E_\gamma(m_{h_c}) \) is the most probable transition energy.

\( B(M) \) denotes the function which is used to describe the background shape. For exclusive decay modes, polynomial functions are used. For \( \eta_c \to 2(\pi^+\pi^-\pi^0) \), the backgrounds are represented with polynomial functions of third order. For other exclusive decay modes, the backgrounds are linear functions. For the inclusive decay mode, it is a combination of the distribution from \( h_c \) sidebands and a polynomial function.

Figure 3 shows the simultaneous fit results. The fitted BFs are summarized in Table III, together with the detection efficiencies and signal yields at each energy point.

**VI. CHARGED TRACK MULTIPLICITY OF \( \eta_c \) INCLUSIVE DECAYS**

The MC simulation for the inclusive \( \eta_c \) decay has been introduced in Sec. III. The performance of the inclusive simulation, to some extent, can be investigated by the consistency of the charged track multiplicity \([10,20,21]\).

Below, we introduce how to obtain the true charged track multiplicity of \( \eta_c \) inclusive decay. An even number of charged tracks is generated in an event due to the charge conservation, while any number of charged tracks can be observed due to the detector acceptance and reconstruction efficiency. The observed charged track multiplicity of \( \eta_c \) can be obtained by fitting for the \( \eta_c \) signal in the \( \pi^+\pi^-\gamma \) recoil mass with the number of extra candidate tracks required to be 0, 1, 2, 3, \cdots, respectively. To obtain the charged track multiplicity at the production level, an unfolding method is employed based on an efficiency
TABLE III. Detection efficiencies ($\epsilon$) for $\eta_c$ inclusive and exclusive decays, fit results including the observed number of signal events ($N_{\text{obs}}$), and the fitted BFs for the four $\eta_c$ exclusive decay modes. The statistical uncertainties of the observed numbers of the signal yields for the inclusive decay are obtained directly from the fit, while the numbers of signal events for the exclusive decays are calculated via Eq. (3) rather than being obtained directly from the fit, so no uncertainties are provided.

<table>
<thead>
<tr>
<th>Decay modes</th>
<th>$\sqrt{s}$(GeV)</th>
<th>$\epsilon$ (%)</th>
<th>$N_{\text{obs}}$</th>
<th>BF (%)</th>
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</thead>
<tbody>
<tr>
<td>$\eta_c \to K^+K^-\pi^0$</td>
<td>4.23</td>
<td>15.95</td>
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<td>4.26</td>
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<tr>
<td></td>
<td>4.36</td>
<td>18.82</td>
<td>30.6</td>
<td></td>
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<td>4.42</td>
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<td>4.42</td>
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<td>Inclusive decays</td>
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<td>8314 ± 584</td>
<td></td>
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<tr>
<td></td>
<td>4.26</td>
<td>45.17</td>
<td>6651 ± 499</td>
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<tr>
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<td>4.36</td>
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<td>6420 ± 420</td>
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</tr>
<tr>
<td></td>
<td>4.42</td>
<td>46.69</td>
<td>11083 ± 615</td>
<td></td>
</tr>
</tbody>
</table>

FIG. 4. Normalized distributions of charged tracks multiplicities at the production level in $\eta_c$ decays, summed over all center-of-mass energies. The blue histogram represents results from MC simulation, while black dots with error bar from data. The label 8 on the axis of $N_{\text{charge}}$ means $N_{\text{charge}} \geq 8$.

TABLE IV. The normalized multiplicity of $\eta_c$ at production level with systematic uncertainties.

<table>
<thead>
<tr>
<th>$N_{\text{charge}}$</th>
<th>Normalized values</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.036 ± 0.011 ± 0.007</td>
</tr>
<tr>
<td>2</td>
<td>0.328 ± 0.035 ± 0.043</td>
</tr>
<tr>
<td>4</td>
<td>0.467 ± 0.044 ± 0.064</td>
</tr>
<tr>
<td>6</td>
<td>0.132 ± 0.033 ± 0.022</td>
</tr>
<tr>
<td>$\geq$8</td>
<td>0.037 ± 0.015 ± 0.009</td>
</tr>
</tbody>
</table>

matrix, whose matrix elements, $e_{ij}$, represent the probabilities of an event generated with $j$ tracks being observed with $i$ tracks. The efficiency matrix is determined from the inclusive $\eta_c$ MC samples. The unfolding of data is achieved by minimizing a $\chi^2$ value, defined as

$$
\chi^2 = \sum_{i=1}^{8} \frac{(N_{i}^{\text{obs}} - \sum_{j=0}^{8} e_{ij} \cdot N_{j})^2}{(\sigma_{i}^{\text{obs}})^2},
$$

where the values $N_{i}^{\text{obs}}(i = 0, 1, 2, \ldots)$ are the observed multiplicities of charged tracks in the data sample, $\sigma_{i}^{\text{obs}}$ are the corresponding uncertainties, while $N_{j}(j = 0, 2, 4, \ldots)$ are the true multiplicities of charged tracks at the production level in the data sample. For simplicity, the events with eight or more tracks are considered in a single value, $N_{\geq 8}$, so are the efficiencies, $\epsilon_{\geq 8}$.

Figure 4 shows the charged track multiplicity distribution of inclusive $\eta_c$ decays after combining the data at the four center-of-mass energies. According to Eq. (5), the normalized numerical results are summarized in Table IV.

VII. SYSTEMATIC UNCERTAINTIES

A. Measurement of $BF(\eta_c \to X)$

The systematic uncertainties on the BF measurements for exclusive $\eta_c$ decays from different sources are described below and listed in Table V. The total systematic uncertainty is determined by the sum in quadrature of the individual values, assuming all sources to be independent.

1. MDC tracking and PID

The uncertainty from the tracking efficiency and PID for the two soft pions in the process $e^+e^- \to \pi^+\pi^-h_c$ cancels since the BFs are measured by a relative method, as mentioned in the introduction. We only consider the uncertainty from tracking efficiency and PID of the $\eta_c$ decay products. The involved charged tracks are pions (not including the pions from $K^0_S$ decay), kaons, and protons. Their uncertainties are studied with different control samples, $e^+e^- \to \pi^+\pi^-K^+K^-$ for pions and kaons, $e^+e^- \to p\pi^-\bar{p}\pi^+$ ($e^+e^- \to p\pi^-\bar{p}\pi^+\pi^-$) for protons. The uncertainties from tracking efficiency are 1% for each
pion, and 2% for each kaon or proton. The uncertainties for PID are 1% for each pion, kaon or proton.

2. \( \pi^0 \) reconstruction

The systematic uncertainty from \( \pi^0 \) reconstruction is studied with \( \psi(3686) \to \pi^0 J/\psi \) using \( 1.06 \times 10^8 \psi(3686) \) events and \( e^+ e^- \to o\pi^0 \to \pi^+ \pi^- \pi^0 \pi^0 \) using a data sample of 2.93 fb\(^{-1}\) collected at the \( \psi(3770) \) resonance. The uncertainty as a function of \( \pi^0 \) momentum is determined. The uncertainty from \( \pi^0 \) reconstruction is calculated with the function, according to the momentum distribution of the \( \pi^0 \) in the decays studied.

3. Kinematic fit

The systematic uncertainty from the kinematic fit is estimated by correcting the helix parameters of the charged tracks and the corresponding covariance matrix of the MC simulations to improve the agreement between data and MC simulations. The results with the corrections are taken as the final results since as the MC simulations are more consistent with the data after corrections. The detailed description can be found in Ref. [22]. The helix parameters are extracted from the control samples, \( e^+ e^- \to K^+ K^- \pi^+ \pi^- \) with data sample taken at \( \sqrt{s} = 4.26 \) GeV, and \( J/\psi \to p\bar{p} \pi^+ \pi^- \). The differences in the detection efficiency between the MC samples with and without the corrections are taken as the uncertainties due to the kinematic fit.

4. \( K_S^0 \) reconstruction

The \( K_S^0 \) reconstruction is studied with two control samples, \( J/\psi \to K^{\pm} K^\mp \) and \( J/\psi \to J/\psi K_S^0 K^{\pm} \). The difference in the \( K_S^0 \) reconstruction efficiency between the MC simulation and the data is 1.2% [23], which is taken as the uncertainty due to \( K_S^0 \) reconstruction.

5. MC model

In the MC simulation, the process \( e^+ e^- \to \pi^+ \pi^- h_c \) is modeled with a phase space (PHSP) distribution. In fact, there is a confirmed intermediate state \( Z_c(4020) \) and a potential intermediate state \( Z_c(3900) \), in the \( \pi^+ \pi^- h_c \) final state. The uncertainty caused by the intermediate states is estimated by mixing the MC events including \( Z_c(4020)/Z_c(3900) \) component according to the measured fractions [9,24]. The difference in the detection efficiency is taken as the uncertainty.

For the exclusive \( \eta_c \) decay modes, intermediate resonant states may affect the detection efficiency. MC samples related to \( \eta_c \) multibody decays are generated by sampling according to the invariant mass distributions or mixing the known intermediate states, or changing the decay model used in the MC simulation. The difference in the efficiency with and without intermediate states is taken as the uncertainty.

The uncertainty due to the inconsistency between data and MC simulation on the charged track multiplicity in inclusive \( \eta_c \) decays is estimated based on the multiplicity obtained by the unfolding method mentioned in Sec. VI. The detection efficiency for inclusive decay can also be recalculated with the following formula:

\[
\epsilon_{\text{inclusive}} = \sum_j \left( N_j \sum_i \epsilon_{ij} \right),
\]

where \( N_j \) are the normalized multiplicities in data, listed in Table IV, and \( \epsilon_{ij} \) are the elements of the efficiency matrix in Eq. (5). The differences between this result and the original
Comparison also indicates that the measured recoil mass of the transition pions and the
since the nominal inclusive efficiency is determined by the
hc
by randomly changing the low and high boundaries of the
mass window selection.

6. \( h_c \) mass window

The uncertainty from the \( h_c \) mass window is estimated by randomly changing the low and high boundaries of the \( h_c \) signal region in the ranges of \([3.512, 3.518] \text{ GeV}/c^2 \) and \([3.532, 3.538] \text{ GeV}/c^2 \) and fitting the spectrum with efficiencies estimated in the corresponding intervals. The procedure is repeated for 800 times, and the distributions of the fitted BFs follow Gaussian functions. The obtained standard deviations are taken as the uncertainties due to the \( h_c \) mass window selection.

7. Fit procedure

This uncertainty arises from the fit range, the background shape, the mass resolution, the parameters of the \( \eta_c \) resonance, the efficiency curves, and the damping factor.

The uncertainty from the fit range is estimated by randomly changing the lower side in the range of \([2.540, 2.555] \text{ GeV}/c^2 \) and higher side in \([3.200, 3.215] \text{ GeV}/c^2 \) and repeating the fit for 800 times. The root mean square (rms) of the resulting distributions are taken as the systematic uncertainties from the fit range.

The uncertainty due to the assumed background shape in the exclusive modes is estimated by changing the order of the Chebychev polynomial functions. For the inclusive decay mode, the \( h_c \) sidebands need to be considered as well, whose systematic uncertainty is estimated by randomly changing the left and right margins of the lower and upper sidebands and repeating the fit. The procedure is performed 800 times. The left and right margins of the sidebands are changed in the ranges of \([3.496, 3.450] \text{ GeV}/c^2 \), \([3.503, 3.507] \text{ GeV}/c^2 \) and \([3.543, 3.547] \text{ GeV}/c^2 \), \([3.548, 3.552] \text{ GeV}/c^2 \) for the lower and upper sideband regions, respectively.

The distributions of the fitted results follow Gaussian functions, and the standard deviations are taken as the uncertainties from the \( h_c \) sidebands selection. The uncertainty from the polynomial is estimated by changing the order of the polynomial.

The discrepancy between data and MC simulation on detection resolution is estimated by a control sample, \( \psi(2S) \rightarrow \pi^+\pi^-J/\psi, J/\psi \rightarrow \gamma\eta' \), \( \eta' \rightarrow \gamma\pi^+\pi^- \). By fitting the \( \eta' \) signals, we can obtain the mass resolution for both data and MC simulations. We change the mass resolutions according to the result obtained from control sample to refit the \( RM(\gamma\pi^+\pi^-) \). The differences on the BFs with and without changing the mass resolution are taken as the systematic uncertainties.

The \( \eta_c \) resonance parameters are fixed to the world average values in the fit. We change these values by \( \pm \sigma \), and the larger difference is taken as the uncertainty.

The efficiency curves, as shown in Fig.2, change slowly with \( RM(\pi^+\pi^-\gamma) \). We find only a very small change in results when constant efficiencies are used. Therefore, the uncertainties due to efficiencies can be neglected.

The uncertainty from the damping factor is estimated by using an alternative form of the damping factor, which is used in the CLEO’s published paper [25]. The differences between the results with the two forms of damping factor are taken as the systematic uncertainty.

B. Charged track multiplicity

The systematic uncertainties on the charged track multiplicity in \( \eta_c \) inclusive decay from different sources are described below and listed in Table VI. They are estimated in a similar way as introduced in Sec. VII A. The total systematic uncertainty is determined by the sum in quadrature of the individual values, assuming that all the sources are independent.

1. MDC tracking and PID

The uncertainties from MDC tracking and PID are the same as those in the measurement of BF(\( \eta_c \rightarrow X \)).
TABLE VII. Measured BFs of $\eta_c \rightarrow K^+ K^- \pi^0$, $K^0_S K^+ \pi^-$, $2(\pi^+ \pi^- \pi^0)$, and $p \bar{p}$ with statistical (the first ones) and systematic (the second ones) uncertainties. The third uncertainties in the results from Ref. [6] are the systematic uncertainties due to the uncertainty of $BF(\psi(3686) \rightarrow \eta \eta_c) \times BF(\eta_c \rightarrow \gamma \eta_c)$. The combined results from PDG are listed in the last column, among which $BF(\eta_c \rightarrow K \bar{K} \pi)$ is provided.

<table>
<thead>
<tr>
<th>Final states</th>
<th>BF (%)</th>
<th>BF (%) from Ref. [6]</th>
<th>BF (%) from PDG [7]</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K^+ K^- \pi^0$</td>
<td>1.15 ± 0.12 ± 0.10</td>
<td>1.04 ± 0.17 ± 0.11 ± 0.10</td>
<td>7.3 ± 0.5 (K \bar{K} \pi)</td>
</tr>
<tr>
<td>$K^0_S K^+ \pi^-$</td>
<td>2.60 ± 0.21 ± 0.20</td>
<td>2.60 ± 0.29 ± 0.34 ± 0.25</td>
<td></td>
</tr>
<tr>
<td>$2(\pi^+ \pi^- \pi^0)$</td>
<td>15.3 ± 1.8 ± 1.8</td>
<td>17.23 ± 1.70 ± 2.29 ± 1.66</td>
<td>17.4 ± 3.3</td>
</tr>
<tr>
<td>$p \bar{p}$</td>
<td>0.120 ± 0.026 ± 0.015</td>
<td>0.15 ± 0.04 ± 0.02 ± 0.01</td>
<td>0.152 ± 0.016</td>
</tr>
</tbody>
</table>

2. $h_c$ mass window


3. MC model

The uncertainty due to MC model mainly comes from the potential $Z_c$ intermediate states and the simulation of the $\eta_c$ inclusive decays. The uncertainty caused by the former has been introduced in Sec. VII A 5. For the latter, the simulation of the $\eta_c$ inclusive decays has been mentioned in Sec. III. To estimate the uncertainty caused by this simulation, we made a comparison on the detection efficiencies with and without removing the unknown decay modes generated with LUNDCHARM. Conservatively, the corresponding difference is taken as the uncertainty caused by the simulation of $\eta_c$ inclusive decays.

4. Fit

The uncertainties due to the fit to the recoil mass spectra of $\pi^+ \pi^- \gamma$ are evaluated by varying the fit range, sideband ranges, mass resolution, resonant parameters of $\eta_c$, and damping factors used in the fit, in similar ways as introduced in Sec. VII A. The spreads of the results obtained with the alternative assumptions are used to assign the systematic uncertainties.

VIII. SUMMARY

In summary, with the data samples collected at $\sqrt{s} = 4.23, 4.26, 4.36$, and 4.42 GeV, by comparing the exclusive and inclusive decays of $\eta_c$, we determine the BFs for $\eta_c \rightarrow K^+ K^- \pi^0$, $K^0_S K^+ \pi^-$, $2(\pi^+ \pi^- \pi^0)$, and $p \bar{p}$ via $e^+ e^- \rightarrow \pi^+ \pi^- h_c$, $h_c \rightarrow \gamma \eta_c$. The results are presented in Table VII; they agree with previous measurements by BESIII [6] within uncertainties, while the accuracy of these BFs is improved. With this improved accuracy, the measurements of the $M1$ transitions of $J/\psi \rightarrow \gamma \eta_c$ and $\psi(3686) \rightarrow \gamma \eta_c$ can be more precise, since such measurements provide combined results of $BF(J/\psi(3686) \rightarrow \eta \eta_c) \times BF(\eta_c \rightarrow \gamma \eta_c)$. Moreover, the charged track multiplicity of $\eta_c$ inclusive decay at production level is quantitatively presented for the first time in Table IV. The good consistency between data and MC simulation for this charged track multiplicity indicates that the current MC simulation works generally well. With this charged track multiplicity, many studies with $\eta_c$ in the final state [26] are possible with higher precision than previously.

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