Optimal Path Smoothing using Clothoids for Autonomous Vehicles in the Quarry Environment

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Abstract

Autonomous vehicles are a rapidly growing field and will play an important role in future society. The introduction of autonomous machines in the quarry environment increases safety in the hazardous working conditions and reduces the risk of accidents caused by human error. By making the machine electric and autonomous it will also improve the fuel efficiency and reduces carbon emission.

The results of this thesis are developed within the electric site project which is lead by Volvo Construction Equipment in Eskilstuna, Sweden. The goal is to transform the quarry and aggregates with the help of the introduction of electric hybrid autonomous machines which is predicted to result in reduced carbon emission by up to 95%. This thesis address the problem of path planning noisy curvatures of the paths that are calculated using coordinates recorded from a manual driven vehicle. This makes the control algorithm for the autonomous vehicle hard to implement. The purpose in this thesis is to develop a procedure to describe paths and smooth those so the curvature becomes smooth and drivable for an autonomous load carrier.

To smooth the paths, we use the concept of clothoids which are curves with linear varying curvature. Clothoids are widely used in road design because of its smoothness properties. We solve the smoothing problem by formulating it as an optimization problem where we minimize the number of clothoid segments to avoid unnecessary noisy curvature variations.

The results are successful and we are able to produce feasible paths for two test tracks Volvo Construction Equipment uses. We also create a Graphical User Interface in MATLAB to facilitate the creation of the clothoid-based paths from recorded GPS coordinates.
Sammanfattning

Autonoma fordon är ett område som har fått mycket uppmärksamhet inom forskningen och kommer ha en viktig roll i samhället i framtiden. Introduktionen av självkörande fordon i öppna gruvmiljöer ökar säkerheten för personer som jobbar i dessa kritiska områden och reducerar olyckor från mänskliga mister. Genom att låta maskinerna vara elektriskt drivena kommer också effektivisera bränsle användningen och minska koldioxidutsläppen.

Resultaten som har tagits fram i denna uppsats är i samarbete med Volvo Construction Equipment i Eskilstuna, Sverige inom projektet Electric Site. Målet är att transformera den öppna gruvindustrin som använder dieselfordon till att använda självkörande och elektriska fordon vilket ska kunna reducera koldioxidutsläppen med upp till 95 %. I uppsatsen tar vi upp problemet med banplanering av brusig kurvatur från banor som är inspelade från en GPS på en manuell körende bil. Syftet med denna uppsats är att utveckla ett system som kan beskriva banor och slita upp dessa sådan att kurvaturen blir mjuk och autonomt körbara för en lastare.

För att slita ut banorna använder vi klotoider, som är kurvor med linjärt varierande kurvatur längs kurvan. Klotoider är ett välansvikt koncept inom vägdesign på grund av mjuka kurvaturesegenskaper. Vi löser utslätningsproblemet genom att formulera det som ett optimeringsproblem där vi minimerar antalet klotoidsegment för att undvika onödiga kurvaturevariationer i form av brus.

Resultaten är framgångsrika och vi lyckas skapa körbara banor för två testbanor som Volvo Construction Equipment använder. Vi skapar också ett grafiskt användargränsnitt i MATLAB för att underlätta användare att skapa klotoidbaserade banor från GPS-koordinater som spelats in.
Acknowledgement

I would like to thank my supervisor Johan Sjöberg at Volvo Construction Equipment for taken his time to plan, engage, guide and assist me during this thesis period. I am grateful that Volvo gave me this opportunity to contribute to a such interesting and useful project which have been valuable for me. Everyone involved in this big project have been very friendly and helpful. This have contributed to have great working environment which I appreciate. Finally, I would like to thank Johan Karlsson, my supervisor at KTH for the support and for given me valuable guidelines and tips during the whole thesis period.
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1 Introduction

Autonomous vehicles is a rapidly growing field and is already a reality in our society. The enormous increase of demand for transportation in our current society creates congestions as well as safety and environmental concerns. These challenges have raised goals from world institutions to optimize road utilization and efficient fuel consumption \[1\]. Currently, it has been estimated that the traffic congestions cost Europe about 1\% of the European Union’s gross domestic product annually \[2\].

The emergence of autonomous vehicles are here to fulfill and solve these problems. In the past years, many major manufactures and technology companies have started to collaborate in projects around autonomous vehicles. The introduction of autonomous vehicles will solve many of the current traffic challenges we face but in the same time also introduce complications of the traffic laws. Who should be responsible in a traffic accident when the vehicle is autonomous? Which decision is the correct decision make when the autonomous car is faced to choose from two bad scenarios? These are some of the questions we should ask ourselves and are the most challenging to legislate and involves ethical aspect. However, the main focus of this thesis will be the technicalities of the path design for the autonomous vehicles in the quarry environments.

In this section, we motivate the need for autonomous vehicles, especially in the quarry environment. We follow this up with the objective of this thesis. Then, we give a brief introduction of the problem formulation in general we want to tackle in this thesis. Finally, we provide the outline for the rest of this thesis.

1.1 Motivation

With the advancement in technology in the recent years and the increased demand for economical growth and environmental awareness, it has put pressure on the transportation sector to innovate and provide solutions to reduce fuel consumption, increase efficiency in road utilization and goods transportation. In construction sites, mining areas and loading terminals, workers are facing dangerous environments and tedious task everyday. In combination with the increasing demand of products, the production sites are moving to remote areas and thus experiencing more harsh working conditions. Furthermore, many enterprises are facing high administrative cost due to increased regulations regarding to environmental compliances. These challenges give incentives to the development of autonomous vehicles. Autonomous vehicles will tackle tougher safety requirements and provide a more efficient fuel consumption. Traffic congestions will also be reduce since the autonomous vehicles are able to drive closer to each other. All these solutions will result in economical and financial benefits.

Autonomous vehicles will play an important role in the future for the mining industry. Many challenges such as fuel consumption, working efficiency, environmental issues and safety issues can be solved with the introduction of autonomous vehicles. With autonomous vehicles we can for instance optimize driving in order to reduce emissions, if we also introduce electric driven vehicles the emissions can be further reduced. Another major advantage with autonomous vehicles in the mining environments is the increase of working effi-
ciency and safety for the sites. The lack of driver compartment also reduce the weight of the vehicles which allow them to have a different design structure to carry heavier loads. With the introduction of this technology we could eliminate injuries and fatalities caused in the quarry sites and reduce the usage of fossil fuel.

1.2 Objective

The problem to tackle in this thesis is to smooth a path with given space co-ordinates recorded from HX1, Volvo’s load carrier. This smoothed path is then loaded into the HX1’s system in order for it to execute the path with a given control algorithm. The HX1 is equipped with necessary sensing technologies to allow it to self-localize accurately. We intend to design a smoothing algorithm with constraints on max curvature the HX1 can handle and an error within a certain tolerance. A more detailed description of the HX1 can be found in section 2.3.

![Illustration of path smoothing](image)

Figure 1.1: Illustration of linearly interpolated way-points with its clothoid fit segments and the maximum error allowance.

Here the black lines denote the linearly interpolated way-points, the red curve consist of several clothoid segments where the square points denotes the kink-points (more of that in the theory section). Finally, the blue curve denotes the error the path is allowed to have.

1.3 Problem formulation

The approach used in this thesis centers around clothoids. We use clothoids due to its properties of linear variation of the curvature along with the path arc length. This produces low lateral jerk and therefore more drivable path for the HX1. A more detailed explanation of clothoid can be found in section 4.2.

In short, the problem formulation that we address in this thesis is to solve the following optimization problem:
1.4 Thesis Outline

Section 2 Background, we give the reader a solid overview of the autonomous vehicles and their advantages. We provide for example in the subsections 2.1 and 2.2 a brief history of quarry and autonomous vehicle respectively. In the following subsections we provide background such as general classification of the automations levels and an overview of the challenges autonomous vehicles face. We also offer the reader an insight of Volvo’s Electric Site project and an introduction of the HX1 load carrier which is the vehicle we are going to work with throughout in this thesis.

Section 3 Related work, will provide the reader a few other projects that had similar objectives and present reader with a brief description of methods and results from these projects. It also in-cooperates the most recent stage of the development in path smoothing ideas.

Section 4 Theoretical background, provide the reader with the relevant theory for the thesis used in the method section later, in order to understand the why and how our task was performed.

Section 5 Method, in this section we provide the reader with information regarding how the methods were implemented.

Section 6 Experimental results and discussion, provides the results in terms of numerous plots to give the reader more visual understanding of what we are able to achieve. We will also discuss and comment the results we used and achieved.

Section 7 Conclusions and future work, provides the final conclusions of the project and future improvements for the models and topic we have touched on in this thesis.

Section 8 Appendix contains various plots and screen shots of the experimental results that were not include in the Results section.
2 Background

The idea of self driving cars can be dated back as early as 1950s with the introduction of proportional cruise control. With the breakthrough 30 years later in microprocessor technology the automation in the transportation industry took a big lift [3].

The rapid climate change is one of the biggest challenges our society faces. To reduce the global average temperature increase requires improvements and development in low carbon-emission innovations in industries and sectors among other areas [4]. For the transportation industry, electric cars have been identified as a major player for the reduction of future carbon emissions, greenhouse gases and energy consumption [5].

The use of electricity as the primarily power source in vehicles have several advantages. For instance, they have a higher efficiency and lower operating cost compared to the traditional fossil fuel based power source with combustion engines [6]. As a result, this leads to other advantages such as less air pollution, vibration and noise. There will also be a lower maintenance cost and increased reliability [7]. The progression for the electrification of vehicles have been in parallel with the introduction of autonomous vehicles. Today, almost all of the major car manufactures offers either a full battery electric driven or hybrid electric driven car [8][9]. As mentioned in the motivation section, the combination of electrification, automation and wireless connections adds value for the users and businesses as well as benefits in performances. Autonomous vehicles that are connected are able to choose routes for its destination in order to minimize the energy consumption and ensure the best usage of its power capacity for a given road profile.

In the construction site environments, which is the focus in this thesis, the vehicles perform for the most part repetitive manoeuvres and cooperate with other machines to accomplish common tasks. This more static environment with pre-known road profile and paths requires easier planning to optimize the energy consumption. The connection between the vehicles in, for example, a quarry environment to complete common goals have created the idea of a global fleet management system where the system are assigned to perform the energy optimization and productivity for the entire site.

Since year 2015, the five largest construction equipment manufacturers Caterpillar, Hitachi, Komatsu, Terex and Volvo have shown huge interests in developing this system to make this a reality [10][11][12][13][14]. The majority of them have also drafted plans for the development of autonomous equipments for the commercial market [15].

As in 2017, Caterpillar announced there expansion plan for the autonomous trucks in the commercial market. In Australia, their autonomous trucks have shown 20 % more efficiency than traditional trucks. Komatsu launched a system called Frontrunner, which is a Autonomous Hauler System (AHS). The AHS have been proven to have several advantages both from an environmental stand point as well as from a business and safety perspective. These include reduced CO\textsubscript{2} emissions, reduced operating cost, increased safety on mining sites, reduced number of drivers working in hostile and remote conditions and increased productivity and efficiency [16][17]. Hitachi have also introduced an Autonomous Hauler System which they claim to increase flexibility, improve safety and operation efficiency for their customers [18]. Volvo have also similar
plans, they presented their Electric Site project and predicts to cut the \( \text{CO}_2 \) emissions by 95% as well as reduce the total ownership cost with 25% [19].

### 2.1 Brief introduction and history of quarrying

A quarry is one type of an open-pit-mine, where the extracted materials are stones for building materials. The most common type of quarry materials are dimension stones and aggregates. The methods used since the beginning of extraction of stones and other materials from quarries have changed dramatically. In the beginning, building materials that were extracted from the quarries was carried or dragged out of the quarries manually [20]. Today, workers utilize mechanical tools to extract and transport materials from the quarries. These includes drilling equipments, blasting equipments and hauling equipments. The majority of these equipments are powered by diesel fuel and require huge amount each year. A reduced usage of this type of power source could have a severe impact on the environment.

### 2.2 Brief history of autonomous vehicles

The autonomous vehicles field was taken to the spotlight for the first time around 1980 to 1990. Several projects in the U.S and Europe regarding this field was presented, such as the Navlab vehicle by Carnegie Mellon University [21] and the UniBw Munich and Daimler Benz’s autonomous driving at PROMETHEUS [22]. For the first time, vehicles were able to perform lane changing and vehicle tracking in a three lane French Autoroute with speeds up to 130 km/h. This achievement was a huge breakthrough for autonomous driving. Around the same time, the California partners for advanced transportation technology (PATH) program also developed concepts for automated highway systems [23][24][25]. The main goal for the PATH project was to display technical feasibility rather than to used it for commercial products. One of the most eminent project from PATH was to demonstrated the possibility of vehicle platooning at highway speed, a field that is highly researched at present by Scania [26]. The idea is to group vehicles into platoons, this allow the vehicles to drive close to each other with high speed in a convoy. There are several advantages with this concept, one is the reduction of aero-dynamical forces on the vehicles which results in reduction of fuel consumption. This can be put in comparison with road cycling platoons. Another advantage is increase of capacity on the roads due to the fact that the vehicles are able to stay much closer to each other with continuous wireless communication.

In the beginning of the 21st century, a Stanford robot, named Stanley won the DARPA challenge by completing an unrehearsed 230 km long route through the Mojave desert in less than 7 hours, autonomously [27]. A couple of years later, DARPA announced a new challenge involving moving traffic and obstacle avoidance in traffic situations [15].

In recent years, projects like PROUD, an abbreviation for Public Road Urban Driver test, have merged. The project was aimed to test an autonomous vehicle in a urban environment and highways in Italy. The people behind the project stated it was the first test of it kind where the driverless vehicle was put in operation in public urban roads with roundabouts, pedestrian crossings, freeway junctions, traffic signals and regular traffic [28].
2.3 Automation classification and the journey to full automation

The definition of autonomous vehicle is not so easy to pin-point. What does it mean when, for example, Tesla market their cars as autonomous/self driving vehicles? To illustrate what autonomous vehicles mean we divide them into six major levels [29] [30]

- **No-Automation (level 0):** This is the traditional way of driving, where the driver has complete control of the vehicle’s control inputs such as throttle, brake and steering, in any given time.

- **Function-specific Automation (level 1):** Here, the automation constitute of specific control functions for the vehicle. These can be, for example, cruise control, lane guidance and assistance of braking. The driver however, is fully responsible for overall vehicle manoeuvre and is completely engaged.

- **Combined-function Automation (level 2):** Automation at this level involves control of a minimum of two primary control functions to relieve the driver of those control functions. These can be, for example, adaptive cruise control with lane centering. Drivers are still responsible for the control of the vehicle and are still expected to take control at all times, except under certain conditions.

- **Limited Self-driving Automation (level 3):** Under certain traffic scenario and environments the automation enables the driver to cede control of the vehicle functions that are safety-critical under certain traffic scenarios. The driver should be present for occasional control.

- **Self-driving Under Specified Conditions (level 4):** At this level of automation the vehicle can control all the driving critical components under certain conditions.

- **Full Self-driving Automation (level 5):** Automation at this level have full control of the safety critical driving functions such as, throttle, brakes and steering at all time. This only requires the driver to input destination and other navigation inputs for the vehicle but the driver do not need to be available for the critical control at any given time.

The majority of the vehicles that are commercially available today have an automation level between 1 and 3. Two years ago, Tesla launched the model S car [31] which could be considered to be in the level 3 automation category of autonomous cars. Last year, Tesla said the cars will be equipped with hardware that support level 5 automation [32]. However, to reach level 5, there is still a long way to go.

The cars in the society right now have already some of the features of Advanced Driver Assistance System (ADAS) implemented in the system, such as adaptive cruise control and electronic stability systems to name a few. The European Commission has the goal to strive vehicle safety, in order to accomplish that they approach this by search for modern crash technologies. The European Commission’s Cars 21 strategy also envisages the automotive autonomous industry to provide and improve the technology and vehicle safety for occupants and unprotected road users [33].
The main challenges that the autonomous vehicles face constitute of five factors [34]:

- Technology
- Infrastructure
- Legislation
- Consumer market acceptance
- Business incentives

Currently, the autonomous vehicles that are developed are based on sensors and communication based approaches. Until current day, the ADAS that have been developed such as anti-lock braking system, adaptive cruise control, automatic parking and electric stability control [35]. This systems have been introduced to slowly replace and facilitate the driving for human drivers under critical and tedious circumstances. These situations can, for example, result in human losses or inefficient driving. The typical sensors used in ADAS involves stereo cameras and short and long-range RADARs to perceive and respond to information to the environmental dynamics in the surroundings [15]. However, many of the available ADAS technologies have not been proven to be fully reliable due to the fact that it can not accurately recognize and perceive the surroundings as a human. For example, standard and common traffic situations can be repeated and taught by using machines learning and artificial intelligence algorithms. But there are also traffic situations where more complex and unforeseen scenarios can appear and thus difficult for the current available algorithms to process, especially in real time. One disadvantage of the sensor dependent approach is the high-cost when integrating the sensors into commercial vehicles. Despite the decrease of cost of the sensor technology, it is still a high cost barrier for the general public consumers to enter the autonomous vehicle market [15].

To tackle these problems, the incorporation of the connection between the vehicles with each other becomes a potential solution. By increasing the vehicle to vehicle connection (V2V), they are able to cooperate with each other and facilitate for information gathering. One example of V2V implementation is platooning used by Scania [26]. At the same time, vehicles may also connect themselves to the infrastructures (V2I). This will allow vehicles to decentralise the need of standalone equipments such as sensors and complex artificial intelligence technologies [26] [37]. The later proposal also involves heavy investments in infrastructure project in order for this to work in practice but can also reduce road traffic congestions and have other economical benefits for the society [37].

Another aspect to overcome for the autonomous vehicle market is the public acceptance barrier. It has been known for a long period of time that the human mistakes plays an important role of all traffic accidents every year [38]. As a result, an introduction of autonomous driven vehicles can surely reduce the number of traffic accidents. So why does the general public seem reluctant for this technology to take place? One reason can be that the drivers rather want to have the control and ability to avoid an accident in exchange for a slightly higher probability of an accident actually happening. Another reason can be that the drivers want to feel that they are safe, because they have control and familiar with driving, rather actually being safe. In the early stage of the
transition from traditional driving to the autonomous vehicle market, there will be a period, so called transition phase, where driverless vehicles and non autonomous vehicles have to live with each other. This will cause accidents which will result in liability issues. This will change the landscape of how the insurance companies' policies for the transportation business but it will also affect how vehicle manufactures and owners make their choices. Finally, there is also an ethical aspect in all of this. How should the vehicles make tough decisions and which decision is the right decision? For example, what should the vehicle do in a scenario where you have a group of people crossing a bridge and the decision is either to hit them or swerve off the bridge and injure yourself and potentially other road users [39]? Ultimately, it will also change the legislations for the transportation area.

As for transportation in the areas such as taxis, bus and trucks, the introduction of autonomous vehicles can have serious consequences for the drivers of those vehicles. Furthermore, there is also a chance to see a decrease for jobs in maintenance and traffic police. The advancement of the autonomous driving will slowly replace their jobs with machines. This is a harsh reality but one need to remember that the introduction of these vehicles will make the society more efficient and safer. Many of the jobs in the mining industry are often too dangerous and ineffective for a human to perform and therefore the introduction of automation is a necessity to meet a highly populated environment. The introduction of automation in the society has also huge potential from a business perspective, where space saving will be one of the main driving force [35].

2.4 Electric Site

Electric site is a project initiated by Volvo construction equipment emerging technologies department. The aim of this project is to automate and electrify the majority of the vehicles used in the open-pit mining, also called quarry, process. In figure 2.1 we have an overview of the electric site project model. Transportation and movement of rock minerals in the quarry environment are done by two major vehicle types, the hauler and the load carrier.
Figure 2.1: Overview of electric site quarry process. Courtesy: Volvo Construction Equipment.

The haulers are responsible for loading the mineral rocks from the crushers and transport it to the load carriers. When the haulers unload the rocks to the load carriers, the load carriers is responsible for the transportation of the goods to another crusher, automatically, for further process. These procedures and vehicles constitute a network of processes to be automated. The problems in the network involves, smooth path generation and path following for the load carriers. Obstacle avoidance and traffic and fleet control are also aspects one need to consider. The usage of a fully battery-electric driven loader is predicted to reduce the CO$_2$ emissions by 95% and the total ownership cost by 25% [19].

2.5 HX1

HX1 is an autonomous, fully electric driven load carrier. There is a picture of the HX1 in figure 2.2b. This carrier constitutes as one of the elements in the electric site project. This project aims to electrify and automate a transportation stage in a quarry, from excavation to primary crushing and transport to secondary crushing. The machine have a weight of six metric tons but is capable of transporting up to a total load of 15 metric tons of mining materials. The machine has a four wheel drive as well as steering. The intended working speed for the HX1 is set around 7m/s.
Figure 2.2: Two vehicles in the electric site project. Courtesy: Volvo Construction Equipment.
3 Related Work

This section is going to provide a presentation of the methods and researches in the field of path smoothing. Furthermore, related work done in this area. The two mainly used techniques for path smoothing are clothoids and the methods of splines.

In the research [40] conducted by Komoriya and Tanie, they approach the path smoothing problem with B-spline curves. By using b-spline curve they were able to design smooth trajectories in a two dimensional space for a wheel-type mobile robot. The smoothed path passes through specific points with a specified tangent.

In [41], the research was performed by Nelson. The path smoothing problem was tackle by using polar splines. The autonomous guided vehicles desire paths with continuous curvature, by using polar spline this can be satisfied. The use of polar splines also have the advantage of being that it has a closed expression form which provides continuous curvature and can satisfy the boundary conditions at the line-curve points along the path.

In [42] and [43] the use of cubic Bézier curves were in highlight to tackle the path smoothing problem. Where the first one tackled a problem for an mobile robot Yamabico-11 and the later tackled a problem for a non-holonomic robot.

Other researches within this field have been conducted the likes of [44, 45]. In [44] a motion primitive is first used to field the initial feasible path. In [45] and [45] the concept of clothoids have been applied to generate paths with continuous curvatures.
4 Theoretical background

This section is going to contain a theoretical presentation of the methods and models that we are going to use in the thesis. We will start to briefly describe the vehicle kinematics used in this thesis to determine the maximum allowed curvature the vehicle can perform. Next, we present the reader with the concept of clothoid curves used in the road-map design, which is the main concept we are going to use in this thesis for path smoothing. Finally, we present the reader with the theory behind optimization, in particular convex optimization and convexity, which are taken advantage of in this project.

4.1 Kinematics

To obtain the max curvature limit we need to examine the vehicle kinematics. Below we briefly describe two suitable ways of modelling the vehicle dynamics for the HX1.

4.1.1 Bicycle model

At low speed movement for a non-holonomic vehicle we can approximation the dynamics with only its time-domain kinematics. This is because the lateral dynamics have small influence. The kinematic equations are given by

\[
\dot{x} = v(t) \cos(\theta(t)), \\
\dot{y} = v(t) \sin(\theta(t)), \\
\dot{\theta} = \frac{v(t)}{L} \tan(\phi(t)),
\]

where \(x\) and \(y\) are the global coordinates of the vehicle, \(v(t)\) is the longitudinal velocity of the vehicle, \(L\) is the length between the front and rear axle, \(\theta\) is the yaw angle and \(\phi\) is the steering angle for the front wheels.

We can express these equations in term of the space coordinates instead by using the fact that \(dt \cdot v(t) = ds\). In addition we are assuming \(v(t)\) is a continuous function and is not equal to zero. Hence, we get the following

\[
\frac{dx(s)}{ds} = \cos(\theta(s)), \\
\frac{dy(s)}{ds} = \sin(\theta(s)), \\
\frac{d\theta(s)}{ds} = \frac{1}{L} \tan(\phi(s)).
\]
4 THEORETICAL BACKGROUND

4.1.2 Two-steering wheels model

A kinematic model of the a two-wheel steering vehicle at low speed, using the midpoint of the vehicle can by expressed as [46]

\[
\dot{x} = v(t) \cos(\theta(t)), \quad (4.7)
\]

\[
\dot{y} = v(t) \sin(\theta(t)), \quad (4.8)
\]

\[
\dot{\theta} = \frac{1}{L} [v_f \sin(\varphi_f(t)) - v_r \sin(\varphi_r(t))], \quad (4.9)
\]

where \( r \) stands for rear and \( f \) stands for front. We also define \( \varphi_f(t) \) as the steering angle for the front wheel and \( \varphi_r(t) \) as the steering angle for the back wheel. Assume that the vehicle is not slipping, this implies that the velocities on the vehicle's longitudinal axis must be equal. We can express this as follows

\[ v_r \cos(\varphi_r(t)) = v_f \cos(\varphi_f(t)). \]

Now we can rewrite the original model by inserting this equation as

\[
\dot{x} = v(t) \cos(\theta(t)), \quad (4.10)
\]

\[
\dot{y} = v(t) \sin(\theta(t)), \quad (4.11)
\]

\[
\dot{\theta} = \frac{v_f \cos(\varphi_f)}{L} \left[ \tan(\varphi_f(t)) - \tan(\varphi_r(t)) \right]. \quad (4.12)
\]

Furthermore, if the vehicle is symmetrically steered, i.e. \( \varphi_r = \varphi_f \) we can further simplify the model. Let \( \varphi_r = \varphi_f = \varphi \) then

\[
\dot{x} = v(t) \cos(\theta(t)), \quad (4.13)
\]

\[
\dot{y} = v(t) \sin(\theta(t)), \quad (4.14)
\]

\[
\dot{\theta} = \frac{2v_f}{L} \sin(\varphi(t)). \quad (4.15)
\]
We can transform this into the $s$ domain by using $dt \cdot v(t) = ds$. This leaves us with

$$\frac{d\theta(s)}{ds} = \frac{2}{L} \sin(\varphi(s)). \quad (4.16)$$

The left-hand side of the last equation is equivalent to the curvature $\kappa$. This allows us to determine the maximum curvature $\kappa_{\text{max}}$ for the vehicle given its maximum steering angle and vehicle length which we will use in the constraints for the optimization problem later on. In our case the length $L$ between the front and rear wheel axle is $3.2m$ and $\varphi$ is approximately $18^\circ$.

### 4.2 Clothoid

Clothoid, also known as Euler’s spiral is used to find the best transition curve and is widely used in road-map design due to its properties \[47\]. One property is that the curvature $\kappa$ of a clothoid segment is a linear function with respect to the travelled distance $s$. Mathematically, the underlying equations that describe the clothoids are known as Fresnel integrals. It was not until 1890 when Arthur Talbot discovered a direct connection between clothoids and autonomous driving when he derived an equation of clothoid to be a transition curve for the railway tracks \[48\].

![Figure 4.2: Visualization of one segment of a clothoid curve.](image)

In the beginning of path planning for non-holonomic vehicles, the main focus was to generate Dubins’ curves, which was paths obtained by connecting straight lines and circular arcs \[49\]. A disadvantage with these curves are that they do not have continuous curvature. In order for a non-holonomic vehicle to follow these curves, the vehicle must stop and re-orientate the steering angle \[50\]. Otherwise, the discontinuities make it impossible for path tracking to be accurate in position and orientation since instantaneous changes in the steering mechanism is needed, which is impossible. This complication formed the emergence of smooth path planning, which meant that path planners produced paths with methods like B-splines, polar splines, cubic splines and clothoids \[15, 51\]. The main reason we did not use splines in this thesis is the expressions and
functions become easily complex. Clothoids on the other hand have the easiest curvature function expression and at the same time provide smooth transition curves, which are desired [45, 52].

The general description of a clothoid curve in parametric form is given by:

\[
\begin{align*}
  x(s) &= x_0 + \int_0^s \cos(\theta(\zeta))d\zeta, \\
  y(s) &= y_0 + \int_0^s \sin(\theta(\zeta))d\zeta, \\
  \theta(s) &= \theta_0 + \int_0^s \kappa(\zeta)d\zeta, \\
  \kappa(s) &= \kappa_0 + cs,
\end{align*}
\]

where \(x(s)\) and \(y(s)\) represent the Cartesian coordinates of the curve, whereas \(\theta(s)\) defines the angle by the tangent along the path (yaw angle). Here we can see that the curvature \(\kappa\) is a linear function with respect to the travelled distance \(s\) and varies linearly with the constant change rate of \(c\).

The connection point between the end of one clothoid segment and the start of the following clothoid segment is called a kink-point. Clothoid give us an appropriate transition curve which help to reduce the distance between two path sections. Other advantage using clothoids is the linear change in the steering angle. The curvature changes are linear with the path arc-length, this leads to minimized shock in the connection between the straight lines and circular arcs. Furthermore, clothoidal road-map designs give a linear variation in the vehicle's steering angle and make the steering much simpler and prevents sudden changes in lateral acceleration, which in turn lower the lateral jerk [53]. This is a property we prefer and is also one of the main reason we use the clothoids for path smoothing. It is also easier to design a control algorithm for the steering to be implemented while travelling along clothoid turns.

### 4.3 Convex Optimization / Convexity

An important aspect to consider in optimization is to formulate the optimization problem on linear form. This allows for less complex calculations and more robust convergence to global optimality. One want to model the real world scenario accurately but at the same time on linear form. So a great deal in optimization is to find smart ways to formulate the objective function and the constraints on linear form.

Here we will present the concept of convexity and the result of convexity for optimization problems. Consider the optimization problem

\[
\begin{align*}
  \min & \quad f_0(x) \\
  \text{subject to} & \quad f_i(x) \leq b_i, \quad i = 1, ..., n
\end{align*}
\]

this problem is a convex optimization problem if \(f_0(x), ..., f_i(x) : \mathbb{R}^n \rightarrow \mathbb{R}\) satisfy the following inequality

\[
f_i(\alpha x + \beta y) \leq \alpha f_i(x) + \beta f_i(y),
\]

for all \(x, y \in \mathbb{R}^n\) and where \(\alpha, \beta\) are positive numbers \(\in \mathbb{R}\) and \(\alpha + \beta = 1\) [54]. One of the major advantage with convexity is that any local optimality automatically gives global optimality [54].
4.3.1 Convex set

A set $S$ is a convex set if for any points $x, y \in S$ and any $\lambda \in [0,1]$ we have

$$(1 - \lambda)x + \lambda y \in S.$$  

(4.23)

This means that $S$ is a convex set if one can draw a straight line between two points in the set and every points on that line also remains in the set.

4.3.2 Linear programming

Linear programming problems are a subclass of the convex optimization problems. This problem have to be linear in its objective function and its constraints and can be formulated in standard form as [54]

$$\begin{align*}
\min \quad & c^T x \\
\text{subject to} \quad & Ax = b, \\
& x \geq 0,
\end{align*}$$  

(4.24)

where $c, x \in \mathbb{R}^n, b \in \mathbb{R}^m$ and $A \in \mathbb{R}^{m \times n}$. This is usually called the primal problem. There is also a dual problem on standard form related to this primal problem and is expressed as

$$\begin{align*}
\max_{y,s} \quad & b^T y \\
\text{subject to} \quad & A^T y + s = c, \\
& s \geq 0,
\end{align*}$$  

(4.25)

where $y \in \mathbb{R}^m$ and $s \in \mathbb{R}^n$ is a slack variable. The variable $y$ is the solution to the dual problem and is the Lagrange multiplier to the primal problem. The duality gap is defined by

$$c^T x - b^T y$$  

(4.26)

and is a quantity $\geq 0$. The duality gap is 0 at $(x^{opt}, y^{opt})$. The termination criteria for interior linear programming methods are commonly based on the value of the duality gap.

4.3.3 Optimality conditions

The optimality conditions or the KKT conditions are the following

$$\begin{align*}
Ax &= b, \\
A^T y + s &= c, \\
XSe &= 0, \\
(x, s) &\geq 0,
\end{align*}$$  

(4.27, 4.28, 4.29, 4.30)

where $e$ is a vector with ones and $X$ is defined as
This notation can be used to define $S$. The equation (4.27) is commonly known as primal feasibility, equation (4.28) is known as dual feasibility and equation (4.29) is known as complementarity. Let $\mathbf{x}^* = (x_{opt}, y_{opt}, s_{opt})$, this is called the primal-dual solution [55].

### 4.3.4 Primal-dual interior-point method

Here we going to present one method, primal-dual variants, in the primal-dual interior-point family. This is the method the solver in CVX we use in this thesis. In order to find the the primal-dual solution $\mathbf{x}^*$ the method uses variants of Newton’s method to satisfy the conditions (4.27-4.29) and modifies the step lengths as well as the search directions to fulfill condition (4.30) in every iteration [56]. Let us define

$$ F(\mathbf{x}, \mathbf{y}, s) = \begin{bmatrix} \mathbf{A}^T \mathbf{y} + s - \mathbf{c} \\ \mathbf{Ax} - \mathbf{b} \\ \mathbf{XSe} \end{bmatrix} $$

(4.32)

and rewrite the optimality conditions as $F(\mathbf{x}, \mathbf{y}, s) = 0, (\mathbf{x}, s) \geq 0$. Let $\delta = (\delta x, \delta y, \delta s)^T$ be the search direction. Lets for simplicity define $\mathbf{x}^{(k)} = (x^{(k)}, y^{(k)}, s^{(k)})$. The Newton’s method involves to solve

$$ F(\mathbf{x}^{(k)})\delta = -\nabla f(\mathbf{x}^{(k)}) $$

(4.33)

around the current iterate $(\mathbf{x}^{(k)})$ and gives a search direction $\delta$ if $F(\mathbf{x}^{(k)})$ is positive definite. If the iterates are strictly feasible then equation (4.33) becomes

$$ \begin{bmatrix} 0 & \mathbf{A}^T & \mathbf{I} \\ \mathbf{A} & 0 & 0 \\ \mathbf{S} & 0 & \mathbf{X} \end{bmatrix} \begin{bmatrix} \delta x \\ \delta y \\ \delta s \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\mathbf{XSe} \end{bmatrix} $$

(4.34)

The next iterate is given by

$$ (\mathbf{x}^{(k+1)}) = (\mathbf{x}^{(k)}) + t_k \delta^{(k)} $$

(4.35)

where $t_k \in (0, 1]$ is the line search parameter and should be chosen so it satisfies

$$ f(\mathbf{x}^{(k)}) + t_k \delta^{(k)} < f(\mathbf{x}^{(k)}) $$

(4.36)

### 4.3.5 Primal-dual interior-point algorithm

The equations in (4.34) satisfies the first order optimality conditions for the problem in (4.24) [56]. The Newtons equations for the logarithmic barrier reformulation of the primal problem are [57]

$$ \begin{bmatrix} 0 & \mathbf{A}^T & \mathbf{I} \\ \mathbf{A} & 0 & 0 \\ \mathbf{S} & 0 & \mathbf{X} \end{bmatrix} \begin{bmatrix} \delta x \\ \delta y \\ \delta s \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\mathbf{XSe} + \tau \gamma \mathbf{e} \end{bmatrix} $$

(4.37)
where $\tau \in [0, 1]$ and $\gamma = \frac{x^T s}{n}$ is the average value of the complementarity products, also called the duality measure. For further details on logarithmic barrier term see [56]. The general primal-dual interior point method is [55].

**Algorithm 1** General primal-dual interior point algorithm

1: **procedure**
2: Choose $x^{(0)} := (x^{(0)}, y^{(0)}, s^{(0)})$ so it is feasible
3: Let $k = 0$
4: repeat:
5: Choose $\tau^{(k)} \in [0, 1]$ and $\gamma^{(k)} = \frac{x^T s}{n}$
6: Compute the primal-dual Newton directions $\delta^{(k)}$ using (4.37)
7: Let $x^{(k+1)} = x^{(k)} + t_k \delta^{(k)}$, and choose $t_k$ such that $(x^{(k+1)}, s^{(k+1)}) > 0$
8: Let $k = k + 1$
9: until Convergence

### 4.4 $l_p$-norms

The $l_p$"norm" is defined by the number of non-zero elements in a vector $x \in \mathbb{R}^n$. The $l_1$-norm is defined by

$$
||x||_1 = \sum_{k=1}^{N} |x_k|, \quad (4.38)
$$

which is the sum of the absolute values for the components in the $x$. The $l_2$-norm or commonly called the euclidean norm is define by

$$
||x||_2 = \sqrt{\sum_{k=1}^{n} x_k^2} \quad (4.39)
$$
5 Method

In this thesis we want to achieve constant change in curvature along the path, thus giving the HX1 a smooth path to complete. In order to do so, we can formulate it as an optimization problem. Let $\kappa'(s)$ be a piecewise constant function on $[s_0, s_f]$ and let $\rho(\kappa'(s))$ be defined as the number of discontinuities of $\kappa'(s)$ on $[s_0, s_f]$. We seek to minimize $\rho(\kappa'(s))$, thereby finding a solution which consist of as few clothoid segments as possible. We can formulate this as an optimization problem as follows

$$\min_{\kappa} \rho(\kappa'(s))$$
subject to
$$\frac{dx(s)}{ds} = \cos(\theta(s)),$$
$$\frac{dy(s)}{ds} = \sin(\theta(s)),$$
$$\frac{d\theta(s)}{ds} = \kappa(s),$$
$$\left\| \begin{pmatrix} x(s) \\ y(s) \end{pmatrix} - \begin{pmatrix} x_{ref}(s) \\ y_{ref}(s) \end{pmatrix} \right\|_2 \leq \epsilon, \quad \forall s \in [s_0, s_f],$$
$$|\kappa(s)| \leq \kappa_{max}, \quad \forall s \in [s_0, s_f],$$
$$x(s_0) = x_{init}, \quad x(s_f) = x_{final},$$
$$y(s_0) = y_{init}, \quad y(s_f) = y_{final},$$
$$\theta(s_0) = \theta_{init}, \quad \theta(s_f) = \theta_{final},$$
$$\kappa'(s) \text{ piecewise constant.}$$

(5.1)

To solve this optimization problem, we solve an approximated problem by discretizing and linearizing the problem.

5.1 Path sparsification with clothoids

The nodes where the clothoid segments starts and ends are called kink-points. In every kink-point, the curvature changes. As we want to smooth our path we want to represent it few as few clothoid segments as possible. Hence, this is equivalent to minimize the number of the kink-points.

Given a reference path, we can describe it as a set of linearly interpolated way-points containing position information. In this section we describe a method of sparsifying this references path with help of a set of clothoids.

Let the length of the path be $L$ and lets discretize the path with $N$ points $x_i$ where $i \in [1, 2, ..., N]$. Let $\Delta s_i = s_{i+1} - s_i$ be the distance between two way-points and let the starting point be given as $[x_1, y_1, \theta_1]$. We use Euler forward difference to discretize the positions for the parametric from of the clothoid. Then we will obtain the following


\[x_1 = x_{\text{init}}\]
\[x_i = x_{i-1} + \Delta s_{i-1} \cos(\theta_{i-1}), \quad i \in 2, \ldots, N\]
\[y_1 = y_{\text{init}}\]
\[y_i = y_{i-1} + \Delta s_{i-1} \sin(\theta_{i-1}), \quad i \in 2, \ldots, N\]  
(5.2)

\[\theta_1 = \theta_{\text{init}}\]
\[\theta_i = \theta_{i-1} + \Delta s_{i-1} \kappa_{i-1}, \quad i \in 2, \ldots, N.\]

We can write this on matrix-vectorial form as

\[\begin{align*}
    x & = e_1 x_{\text{init}} + I \Delta x + \Delta s \circ \cos(I \Delta \theta), \\
    y & = e_1 y_{\text{init}} + \text{diag}(\Delta s) I \Delta \sin(\theta) \\
    \theta & = e_1 \theta_{\text{init}} + \text{diag}(\Delta s) I \Delta \kappa
\end{align*}\]

where \(\circ\) denotes the Hadamard product and the cosine function applied element-wise. The vectors and matrix are defined by

\[e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{R}^N, \quad I_\Delta = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 \\ 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \end{pmatrix} \in \mathbb{R}^{N \times N},\]
\[\Delta s = \begin{pmatrix} 0 \\ \Delta s_1 \\ \Delta s_2 \\ \vdots \\ \Delta s_{N-1} \end{pmatrix} \in \mathbb{R}^N, \quad \theta = \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \vdots \\ \theta_N \end{pmatrix} \in \mathbb{R}^N.\]

We can write equation (5.3) in a more compact form as

\[\begin{align*}
    x & = (I - I_\Delta)^{-1} \left[ e_1 x_{\text{init}} + \text{diag}(\Delta s) I_\Delta \cos(\theta) \right], \\
    y & = (I - I_\Delta)^{-1} \left[ e_1 y_{\text{init}} + \text{diag}(\Delta s) I_\Delta \sin(\theta) \right], \\
    \theta & = (I - I_\Delta)^{-1} \left[ e_1 \theta_{\text{init}} + \text{diag}(\Delta s) I_\Delta \kappa \right]
\end{align*}\]

where

\[\kappa = \begin{pmatrix} \kappa_1 \\ \kappa_2 \\ \vdots \\ \kappa_N \end{pmatrix} \in \mathbb{R}^N.\]

We can see that the equations for \(x\) and \(y\) are non-linear. To obtain a convex optimization problem, we want to linearize these equations around \(\kappa = \kappa\). \(\kappa\) is
the approximated curvature vector for the reference path. In order to get $\hat{\kappa}$ we can do the following. By rearranging the terms in (5.2) as

$$x_i - x_{i-1} = \Delta s_{i-1} \cos(\theta_{i-1}), \quad (5.7)$$
$$y_i - y_{i-1} = \Delta s_{i-1} \sin(\theta_{i-1}), \quad (5.8)$$

and then dividing the equation (5.8) by (5.7) we can solve for all $\theta$s. This is given by

$$\tan(\theta_{i-1}) = \frac{y_i - y_{i-1}}{x_i - x_{i-1}} \implies \theta_{i-1} = \arctan \frac{y_i - y_{i-1}}{x_i - x_{i-1}}. \quad (5.9)$$

By plugging in the given reference path coordinates, we obtain all the $\theta$ values. We can then use the equation (5.6) from the discretization of the $\theta$ to calculate the $\hat{\kappa}$.

To express $x$ and $y$ coordinates in terms of the curvature $\kappa$, we plug in the equation (5.6) into (5.4) and (5.5) respectively and we obtain

$$x = (I - I_\Delta)^{-1} \left[ e_1 x_{\text{init}} + \text{diag}(\Delta s) I_\Delta \cos \left( (I - I_\Delta)^{-1} [ e_1 \theta_{\text{init}} + \text{diag}(\Delta s) I_\Delta \kappa] \right) \right],$$
$$y = (I - I_\Delta)^{-1} \left[ e_1 y_{\text{init}} + \text{diag}(\Delta s) I_\Delta \sin \left( (I - I_\Delta)^{-1} [ e_1 \theta_{\text{init}} + \text{diag}(\Delta s) I_\Delta \kappa] \right) \right].$$

To write these expression neater, let’s define the following

$$a_x = (I - I_\Delta)^{-1} e_1 x_{\text{init}},$$
$$a_y = (I - I_\Delta)^{-1} e_1 y_{\text{init}},$$
$$B = \text{diag}(\Delta s) I_\Delta (I - I_\Delta)^{-1},$$
$$\gamma = (I - I_\Delta)^{-1} e_1 \theta_{\text{init}}. \quad (5.10)$$

This gives us the $x$ and $y$ coordinates with respect to $\kappa$ as

$$x(\kappa) = a_x + B \cos(\gamma + B \kappa),$$
$$y(\kappa) = a_y + B \sin(\gamma + B \kappa). \quad (5.11)$$

As we have stated before, these equations are not linear and we need to linearize them in order to apply linear optimization algorithms.

### 5.2 Linearization

We use the first order Taylor approximation to linearize equation (5.11). To do that, let’s first take a look at the general case and then we can apply the same principal for our specific case. The general first order formula for the approximation when involving vectors and matrices is the following

$$f(\bar{\xi}) \approx f(\hat{\xi}) + J(\hat{\xi})(\bar{\xi} - \hat{\xi}), \quad (5.12)$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$, $\bar{\xi}, \hat{\xi} \in \mathbb{R}^n$ where $\hat{\xi}$ are the points we want to approximate around and $J$ is the Jacobian.
\[ J_{ij} = \frac{\partial f_i}{\partial \bar{\xi}_j}. \] (5.13)

Let \( f(\kappa) = \cos(\gamma + B\kappa) \) then if we apply equation 5.12 we obtain
\[ f(\kappa) \approx \cos(\gamma + B\hat{\kappa}) + J(\hat{\kappa})(\kappa - \hat{\kappa}). \] (5.14)

Now we need to find what \( J(\hat{\kappa}) \) is. The Jacobian has the form
\[
J = \begin{pmatrix}
\frac{\partial f_1}{\partial \kappa_1} & \ldots & \frac{\partial f_1}{\partial \kappa_n} \\
\vdots & \ddots & \vdots \\
\frac{\partial f_n}{\partial \kappa_1} & \ldots & \frac{\partial f_n}{\partial \kappa_n}
\end{pmatrix}.
\] (5.15)

To find the elements in the Jacobian, let write the vector \( \gamma \) and matrix \( B \) as follows
\[
\gamma = \begin{pmatrix}
\gamma_1 \\
\gamma_2 \\
\vdots \\
\gamma_n
\end{pmatrix} \in \mathbb{R}^n, \quad B = \begin{pmatrix}
- b_1^T & - \\
- b_2^T & - \\
\vdots & \vdots \\
- b_n^T & -
\end{pmatrix} \in \mathbb{R}^{n \times n}. \] (5.16)

Then the elements in first row in the Jacobian can be written as
\[
\frac{\partial f_1}{\partial \kappa_j} \bigg|_{\hat{\kappa}} = -B_{1j} \sin(\gamma_1 + b_1^T \hat{\kappa}), \quad j \in 1, \ldots, n
\] (5.17)

By following the same pattern, the elements for the Jacobian in row \( n \) can be expressed as
\[
\frac{\partial f_n}{\partial \kappa_j} \bigg|_{\hat{\kappa}} = -B_{nj} \sin(\gamma_n + b_n^T \hat{\kappa}), \quad j \in 1, \ldots, n
\] (5.18)

Now, let’s multiply the Jacobian \( J \) with the vector \((\kappa - \hat{\kappa})\). We obtain the following vector
\[
\begin{pmatrix}
\sum_{j=1}^n B_{1j}(\kappa_j - \hat{\kappa}_j) \sin(\gamma_1 + b_1^T \hat{\kappa}) \\
\vdots \\
\sum_{j=1}^n B_{nj}(\kappa_j - \hat{\kappa}_j) \sin(\gamma_n + b_n^T \hat{\kappa})
\end{pmatrix}.
\] (5.19)

From each row in the vector, we can factor out the term \( \sin(\gamma_i + b_i^T \hat{\kappa}) \) for row \( i = 1, \ldots, n \). Hence, we get
\[
\begin{pmatrix}
\sin(\gamma_1 + b_1^T \hat{\kappa}) \\
\sin(\gamma_2 + b_2^T \hat{\kappa}) \\
\vdots \\
\sin(\gamma_n + b_n^T \hat{\kappa})
\end{pmatrix} \circ \begin{pmatrix}
B_{11}(\kappa_1 - \hat{\kappa}_1) + \ldots + B_{1n}(\kappa_n - \hat{\kappa}_n) \\
B_{21}(\kappa_1 - \hat{\kappa}_1) + \ldots + B_{2n}(\kappa_n - \hat{\kappa}_n) \\
\vdots \\
B_{n1}(\kappa_1 - \hat{\kappa}_1) + \ldots + B_{nn}(\kappa_n - \hat{\kappa}_n)
\end{pmatrix},
\] (5.20)

where \( \circ \) denotes the Hadamard product. We can rewrite the first vector more compactly by factor out the sin function and use the definitions in 5.16.
\[
\sin(\gamma + B \kappa) = \begin{pmatrix}
\sin(\gamma_1 + b_1^T \kappa) \\
\sin(\gamma_2 + b_2^T \kappa) \\
\vdots \\
\sin(\gamma_n + b_n^T \kappa)
\end{pmatrix}, \tag{5.21}
\]

using the same definitions from \textcolor{red}{5.16} the second entity can be rewritten as a matrix times a vector.

\[
B(\kappa - \hat{\kappa}). \tag{5.22}
\]

Thus, the linearization becomes

\[
f(\kappa) \approx \cos(\gamma + B \hat{\kappa}) - \sin(\gamma + B \hat{\kappa}) \circ B(\kappa - \hat{\kappa}), \tag{5.23}
\]

where the \circ symbol represent, as before, the Hadamard product. Now let go back and apply the linearization result for our problem in \textcolor{red}{5.11} this will result in the following

\[
x_L(\kappa) = a_x + B \left[ \cos(\gamma + B \hat{\kappa}) - \sin(\gamma + B \hat{\kappa}) \circ B(\kappa - \hat{\kappa}) \right],
\]

\[
y_L(\kappa) = a_y + B \left[ \sin(\gamma + B \hat{\kappa}) + \cos(\gamma + B \hat{\kappa}) \circ B(\kappa - \hat{\kappa}) \right], \tag{5.24}
\]

where we have used the fact that if \(f(\kappa) = B \cos(\gamma + B \kappa)\) and \(B\) is a constant matrix, then the linearization becomes \(f(\kappa) \approx B \left[ \cos(\gamma + B \hat{\kappa}) + J(\hat{\kappa})(\kappa - \hat{\kappa}) \right]\).

5.3 Optimization formulation

The path sparsification and linearization enable us to write the optimization problem on the following discrete form

\[
\begin{align*}
\min_{\kappa} \quad & \|D_2 \kappa\|_0 + \psi \|\kappa\|_1 \\
\text{subject to} \quad & \left\| \left( x^i(\kappa) \ y^i(\kappa) \right) - \left( x^i_{\text{ref}} \ y^i_{\text{ref}} \right) \right\|_2 \leq \epsilon^i, \quad i = 1, \ldots, N \tag{5.25}
\end{align*}
\]

where \((x^i(\kappa), y^i(\kappa))^T\) is the \(i\)th element of the vectors given by equation (5.11), \(\kappa\) is the curvature vector that is to be estimated, \(\psi\) is a scalar to penalise the curvature \(\kappa\), \((x_{\text{ref}}, y_{\text{ref}})^T\) is the reference path, \(\epsilon^i\) is the maximum allowed deviation for coordinate \(i\) from our original reference path, \(\kappa_{\text{max}}\) is the turning angle per meter the HX1 can handle and \(D_2\) the second order difference matrix operator given by

\[
D_2 = \begin{pmatrix}
1 & -2 & 1 \\
1 & -2 & 1 \\
& & \ddots & \ddots \\
& & & 1 & -2 & 1 \\
& & & 1 & -2 & 1
\end{pmatrix}_{(N-2 \times N)} \tag{5.26}
\]
Note for the problem (5.25) that the coordinates to the smoothed path is only dependent on the curvature $\kappa$ of the path. For a piecewise linear function, the second order derivative of the curvature is zero except for in a few point, so called kink-point. By minimizing the second order differences of the curvature $\kappa$ is essentially minimizing the number of kink-points. This results in smooth and linear transitions in curvature change, which is desired. The second term in the objective function penalise the curvature $\kappa$ which makes the path more straight.

In this problem we also want the start and ending point to be the same as the reference path. Furthermore, we want the new path to have the same heading as the original path. In order to accomplish those requirements we set very small values on $\epsilon^i$ for $i = 0, 1, N - 1, N$. This correspond to the initial and the ending point constraints in the optimization problem (5.1).

### 5.3.1 Model 1

The optimization formulation in (5.25) is known as a NP-hard problem, which means it is a complex computational heavy problem that are difficult to solve [58]. This type of problem can be relaxed using the $l_1$-norm, this results in convex relaxations of the original problem [59]. Furthermore, we linearize the constraints using the results from (5.24). This makes our original optimization problem convex, which allow us to solve the problem easily with well known methods.

$$\min_{\kappa} \left\| D_2 \kappa \right\|_1 + \psi \left\| \kappa \right\|_1$$

subject to

$$\left\| \left( \begin{array}{c} x_i^L(\kappa) \\ y_i^L(\kappa) \end{array} \right) - \left( \begin{array}{c} x_i^{ref} \\ y_i^{ref} \end{array} \right) \right\|_2 \leq \epsilon^i, \quad i = 1, ..., N$$

$$\left| \kappa \right| \leq \kappa_{max}.$$  

(5.27)

This optimization problem is a convex optimization problem. To solve the optimization problem in (5.27) we use CVX, an optimization package for solving convex programs [60, 61] in MATLAB.

### 5.3.2 Model 2

In the second model we try to approximate the $l_0$-norm more accurately with the $l_1$-norm. We do that by using an algorithm within the family of the majorization-minimization algorithms. This allows us to approximate the objective function in $l_0$-norm with the $l_1$-norm by performing iteration on the so called convex majorizer function $l_1$-norm [62]. As mentioned in [62], it is important to point out that there are numerous theoretical properties that need to be confirmed and further researched in order to claim any specific properties such as optimality or convergence. The result and the use of this algorithm in this case is analysed from a practical aspect. We rewrite this optimization problem as

$$\min_{\kappa_m} \left\| T_m \circ D_2 \kappa_m \right\|_1 + \psi \left\| \kappa_m \right\|_1$$

subject to

$$\left\| \left( \begin{array}{c} x_i^L(\kappa_m) \\ y_i^L(\kappa_m) \end{array} \right) - \left( \begin{array}{c} x_i^{ref} \\ y_i^{ref} \end{array} \right) \right\|_2 \leq \epsilon^i, \quad i = 1, ..., N$$

$$\left| \kappa_m \right| \leq \kappa_{max}.$$  

(5.28)
where $T$ is a vector in $\mathbb{R}^{N-2}$ with weights that are initially set to 1 and all the inequality constraints are satisfied element-wise. This vector $T$ is updated during each iteration as

$$T_{m+1}^i = \frac{1}{\|D_2 \kappa_m^i\| + \eta}.$$ 

(5.29)

The introduction of the parameter $\eta > 0$ provides stability when we encounter a zero-component. This ensures the zero-valued elements in the vector $(x^i_L, y^i_L)^T$ in the next iteration can take a nonzero estimate. The size of the parameter $\eta$ should be set to be a bit smaller than the expected nonzero elements of $(x^0_L, y^0_L)^T$. As we see, during each iteration the vector $T$ is normalized so that the weights sum up to the number of samples. This means, the weights update in such a way that the elements of the smoothness term near zero receive a bigger weight, meanwhile the other entries, in our case the kink-points, receive a smaller weight. A higher value of the weight in the optimization pushes down the value of elements, this results in a reweighed $l_1$-norm which is more similar to the $l_0$-norm we are trying to approximate.

### 5.3.3 Model 3

In the third model we use the same optimization formulation as model 2

$$\min_{\kappa_m} \|T_m \circ D_2 \kappa_m\|_1 + \psi \|\kappa_m\|_1$$

subject to

$$\left\| \begin{pmatrix} x^i_L(\kappa) \\ y^i_L(\kappa) \end{pmatrix} - \begin{pmatrix} x^i_{ref} \\ y^i_{ref} \end{pmatrix} \right\|_2 \leq \epsilon, \quad i = 1, \ldots, N$$

$$|\kappa_m| \leq \kappa_{max},$$

(5.30)

where

$$T_{m+1}^i = \frac{1}{\|D_2 \kappa_m^i\| + \eta}.$$ 

(5.31)

But after every completion of the optimization problem we use the optimal $\kappa_m$ as the new point $\hat{\kappa}$ we linearize around.

### 5.4 Implementation in Graphical User Interface

To facilitate the generation of path coordinates we created a Graphical User Interface (GUI) in MATLAB. In the GUI, the original GPS coordinates are uploaded as the input. When the coordinates are uploaded we can plot the original path. Thereafter, we can input the deviation limits for every point from the original path and choose specific points to customize the deviation by marking them. Next, we have the option to set the max curvature $\kappa_{max}$ and penalty $\psi$ for the curvature. Finally, we can run the optimization problem to calculate and plot the clothoid based path and save the coordinates in a txt file. Screen-shots of the GUI can be found in the Appendix.
6 Experimental results and discussion

In this thesis, we have focused on two different types of tracks for smoothing. In the plots below we will illustrate the reference path and the clothoid based smoothed path with the three different models we described in the subsections of section 5.3. The reference path was recorded with GPS installed on the HX1, where the path is located in Eskilstuna at Volvo Construction Equipment’s test track area. We will also illustrate how the solutions converges with different numbers of iterations to approximate the $l_0$-norm with the $l_1$-norm. The results we have obtained used the parameter $\eta = 10^{-2}$ in the weight equation in (5.29). Different values of $\eta$ can be used and the convergence can be seen in the plots 8.1, 8.2 in the Appendix. We also present a plot for the test track with a different values on $\epsilon$ in figure 8.3a in the Appendix as well.

6.1 Test Track 1 with model 1

![Figure 6.1: Reference and clothoid path with error $\epsilon$ and no weight $T$.](image)

In this figure we see the linearly interpolated way points in black and the clothoid-based generated path in red. With only one iteration, $m = 0$, which is a bad approximation of the $l_0$-norm we see that there are a lot of kink-points and thus many clothoid segments. Furthermore, we let the path have a maximum allowed deviation of $\epsilon = 2$ meter from the original path. We can see that the clothoid-base path does not violate the imposed constrains on $\epsilon$.

In figure 6.2 we can see the distance and curvature variations from the reference path. First, we notice that the curvature for the original path in black is very noisy and would make the difficult to steer. We can notice the clothoid path in red on the other hand contains several linearly varying segments which is the property we want to achieve to be able for the vehicle to follow the path.
6 EXPERIMENTAL RESULTS AND DISCUSSION

Due to many kink-point we have many clothoid segments and thus several linear segments in red in the curvature plot. This path is autonomously drivable but with the introduction of weights we can improve this result.

6.2 Test Track 1 with model 2

Figure 6.3: Reference and clothoid path with error $\epsilon$ and weight $T$. 
In model 2 we set the number of iterations to five ($m = 4$). We can see in figure 6.3 that with more iterations, e.g., a better $l_0$-norm approximation we need less kink-points thus less clothoid segments to describe the original path. This result is satisfying and is in line with what the weight should accomplish.

As for the model one, we also set a maximum allowed deviation of $\epsilon = 2$ meter from the original path. We can see that the clothoid-base path does not violate the imposed constrains on $\epsilon$. If we allow the deviations constraints to be larger, the clothoid paths will, as expected deviate more from the reference path and fewer kink-points would be needed. This can be confirmed in the figure 8.3a in the Appendix.

![Figure 6.4: Detailed information for the paths in figure 6.3.](image)

Here the curvature for the original path in black is as expected still very noisy and would make it difficult to steer. But the clothoid path in red on the other hand contains several linearly varying segments which is the property we want to achieve to be able for the vehicle to follow the path easily. We can notice that there are fewer linear segments in figure 6.4 in the curvature plot than the one for model 1. One can interpret every sharp corner in the red line as where a new clothoid segment starts and ends, which is exactly where the kink-points are located. Thus, this leads to a more smooth curve that is autonomously drivable.

We can also examine the second derivatives of the curvature $\kappa$ after the first and final iteration of the optimization problem. In figure 6.5 we see the values of the second derivatives for every element in the $D_2\kappa$ vector. We see that the values for $\kappa_{\text{final}}$, for the most part, are smaller, this is because in every iteration the objective function is pushed down. The peaks for $D_2\kappa_{\text{final}}$ tends also to be narrower than for $D_2\kappa_1$ which means the entries in the vectors are smaller for more values, which is in line with what the weight should do.

Finally, we investigate the convergence of both the optimization variable $\kappa$ and the weight $T$ where we measure the error with the $l_2$-norm. Here, $\kappa_{\text{red}}$ and
$T_{\text{ref}}$ denote the values for $\kappa$ and $T$ after the last iteration. We can see in figure 6.6 that after a number of iterations $i$ both the solutions $\kappa_i$ and the weights $T_i$ tend to converge to the final values of $\kappa_{\text{ref}}$ and $T_{\text{ref}}$. This is a sign that the $l_1$-norm approximation for the optimization problem have converged, which is desirable.

To compare the linearized path with the non linearized (exact) path, we plug in the optimal curvature $\kappa$ from the optimization into equation 5.11. In figure 8.7 in Appendix we can see that the non linearized path is very close to the linearized path.

![Figure 6.5: Second derivatives of the curvature](image)

(a) Difference in $\kappa$ after each iteration with 5 iterations.

(b) Difference in $T$ after each iteration with 5 iterations.

![Figure 6.6: Convergence](image)
6.3 Test Track 1 with model 3
In this section we provide the result when we updated the point \( \hat{\kappa} \) we linearize around, with the optimal \( \kappa \) from each optimization iteration.

![Smoothed path with linearization around \( \kappa_m \) from the previous optimization iteration.](image)

![Distance from reference path. max(deltaX,deltaY)](image)

![Curvature \( \kappa \)](image)

Figure 6.7: In a) we have the reference path, smoothed path and the non-linearized path for Test Track 1, where we linearize around the new \( \kappa_m \) from the previous optimization iteration. Here we approximate the \( l_0 \)-norm with five iterations. In b) we show the distance and curvature variations for the original and the smoothed path in detail.

We see from the figure that the non-linearized path is almost identical to the clothoid based path which is a very satisfying result. This means that the error
from the linearization is reduced compared to model 2. In figure 6.8 we notice that the solution $\kappa$ is converging which is good.

![Figure 6.8: Convergence with updated $\kappa$ as linearization point.](image)

### 6.4 Test Track 2 with model 1

The second track we examined in this thesis are shown below. This track is actually not a commonly used track for Volvo but we want to apply our algorithm to it in order to make sure it can handle various types of paths. The results from the Test Track 2 are similar to Test Track 1. The clothoid-based paths that were generated all satisfies the constraints as imposed. The plots show similar behaviour as seen in Test Track 1. The path with one iteration for the $l_0$-norm approximation has more kink-points than the path with five iterations as we can see in figure 6.11 in the model 2 section.
6 EXPERIMENTAL RESULTS AND DISCUSSION

Figure 6.9: Reference and clothoid path with error $\epsilon$ and no weight $T$.

Figure 6.10: Detailed information for the paths in figure 6.9

6.5 Test Track 2 with model 2

Here we notice there are less kink-points, which follows the same pattern as the case for Test Track 1.
Figure 6.11: Reference and clothoid path with error $\epsilon$ and weight $\mathbf{T}$.

Figure 6.12: Detailed information for the paths in figure 6.11.
Once again we notice that difference for the solution $\kappa$ and the weights $T$ decreases after a number of iterations which is good. In figure 8.8 in Appendix we also show that the non-linearize path lies very close to the linearized clothoid path which indicates that the linearized clothoid path is quite accurate.
7 Conclusion and future work

Autonomous vehicles and automation in general have been a hot topic in the field of research recently. Despite the introduction of autonomous vehicles for the consumer market and making them available to the general public there are still a lot of problems to be solved. In the following sections we state the conclusions and discuss possible future research directions for this thesis.

7.1 Conclusion

The focus in this thesis have been on generating curves with linearly varying curvature with the help of the concept of clothoids for autonomous vehicles in the quarry environment. The use of clothoids provide smoothness and less lateral jerk to the autonomous driving experience. One desired property of the clothoids is the linear variation of curvature along the distance travelled.

We formulated this as an optimization problem and solved it by linearizing the dynamics. We also applied this methodology on two different reference paths provided by Volvo Construction Equipment in Eskilstuna, Sweden. To generate clothoid-based paths we discretized the general parametric description of the clothoids and linearized the expressions around the curvature of the reference path. We then minimized the second derivatives of the curvature which is equivalent to minimizing the number of kink-points, e.g., the start and the end-points of a clothoid segment. The original optimization problem is an NP-hard problem, which is computationally complex to solve. The constraints are also non-linear in the original optimization formulation. Therefore, we approximated the objective function with the linear $l_1$-norm and linearized the constraints. Thus, we approximated the optimization problem with a convex optimization problem which can be solved by using standard methods. The optimization problem can be interpreted as a two stage problem, a partition problem and a smoothing/curve fitting problem. The partition problem helps us to locate where the kink-points should be and the smoothing problem tells us how to fit a curve between the kink-points.

In the first model, a relaxation of the $l_0$-norm to $l_1$-norm was made with no weights. This provided a feasible clothoid based path. In the second model we added weights and iterated the solution. The result were more satisfying where we reduced the number of kink-points and thus made the path more smooth. The error between the linearized path and the non-linearized path was also small. In the last model, we used the optimal curvature $\kappa_{opt}$ from each iteration as the linearization point $\kappa$ we linearize around. This also gave a satisfying clothoid based path as in model two. In this model, the error between the linearized and non-linearized path was even smaller than model 2.

Finally we created a Graphical User Interface in MATLAB. One can upload any reference path with GPS coordinates to be smoothed with different parameter values of choice, such as maximum allowed deviation from the reference path and maximum allowed curvature $\kappa_{max}$ the vehicle can handle. The results are satisfying, especially from the model 2 and model 3. We were able to successfully generate drivable paths for the autonomous vehicle HX1 with sufficient precision for both reference paths.
7.2 Future work

To further develop this work and make it more general, it would be a good challenge to implement this method in three dimensions. In certain quarry environments there are some height differences between the start and end-point. It would be interesting to investigate how much more accurate it would be for a three dimensional model compared to a two-dimensional model. It would also be interesting to investigate how much height difference one needs to make a big difference of using the three dimensional model instead of the two dimensional model.

Another potential improvement to explore is to use another discretization methods with higher order than the Euler forward discretization we used in this thesis. This might improve the accuracy but can take longer time to run. This is a trade off one can choose depending on the preference. Finally, one can also reformulate the optimization problem with a time domain approach with a speed variable or approach the problem with an optimal control problem formulation.
8 Appendix

Figure 8.1: Convergence with different values on $\eta$

Figure 8.2: Convergence with different values on $\eta$
Figure 8.3: In a) we see the reference and smoothed path for Test Track 1 with \( \epsilon = 3 \) meter. In b) we see how the distance and the curvature variations for the original and the smoothed path in detail.
Figure 8.4: In a) we have the difference for the solution $\kappa$ and in b) we have the difference for the weight $T$, for Test Track 1 shown in figure 8.3a with $\epsilon = 3$ meter.
Figure 8.5: GUI with $\psi = 0.2$
Figure 8.6: GUI with $\psi = 1.5$. 
Figure 8.7: Test Track 1 - Comparison between the reference path, clothoid based path and the non-linearized path for model 2.

Figure 8.8: Test Track 2 - Comparison between the reference path, clothoid based path and the non-linearized path for model 2.
References


REFERENCES


