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This paper is an extract from a study investigating the way students’ reasoning and utilization of feedback relate to success and failure in task solving. Sixteen 16-year old students solved a linear function task. They were instructed to use GeoGebra as a mediator and they had the responsibility to choose solution strategies. The results were analyzed using Lithner’s (2008) framework of imitative and creative reasoning together with Shute’s (2008) definitions of formative feedback. Schoenfeld’s (1985) protocol analysis was used to structure the path through solving the task. The results showed that students who were successful in solving the task reasoned creatively and used feedback elaborately. This paper focuses on frameworks and the method of the study.

INTRODUCTION

Two students tried to create a vertical line in the graphic-field of a dynamic software, GeoGebra, by submitting algebraic expressions \( y = mx + c \). Their strategy had been to increase the \( x \)-coefficient until the slope became vertical. “We need a really large one, let’s put in \( y = 100x + 1 \)”…. they performed the activity, interpreted the feedback from GeoGebra, and found that their strategy didn’t work… “Wait, if they are vertical they should not intersect the y-axis at all…..”.

Apparently they predicted incorrectly the result of the activity, they received feedback from the software, and they elaborated on the result of the activity, and drew a correct conclusion. It seems like dynamic software such as GeoGebra offers a guide to the students’ task solving in the sense that they are invited to set up target images for their actions, and the computer’s precise feedback of the action offers possibilities to interpret and elaborate on ideas for subsequent actions. Research has shown, that student discussions often are mathematically shallow when they are solving tasks. One reason may be that in regular teaching, students are not encouraged to create original solution-methods; instead they are guided into rote-learning strategies as they are provided with examples and formulas by instructions (Hiebert & Grouws, 2007; Lithner, 2008).

Considering rote-learning, it’s important to investigate the causes, consequences, and alternatives. In the perspective of reasoning Lithner (2008) defines imitative reasoning (IR), which is related to rote thinking and its opposite, creative mathematical reasoning (CMR), which is characterized as creating original methods of solving a problem which are supported by argumentation anchored in mathematics. A study (Jonsson,
Liljeqvist, Norqvist, & Lithner, 2014) showed students practicing CMR learned better than those who practiced IR. On the assumption that CMR is better for learning Granberg & Olsson (2015) performed a study investigating the way interactive software (GeoGebra) supported CMR. It was found that GeoGebra guided students into creating goals, planning activities, receiving feedback, and evaluating the result of the activity. In the present study, the way of using feedback and the associated evaluation is further investigated through questions about the relationship between the students’ reasoning and their use of feedback generated by GeoGebra.

Therefore the aim of this study is to investigate the relationship between the students’ reasoning and the way students utilize the feedback provided by GeoGebra.

AIM AND RESEARCH-QUESTIONS

The aim of this study is to develop understanding about students´ utilization of feedback from software, associated to their reasoning and success in task solving during joint problem solving aided by GeoGebra.

The research questions guiding this study are:

- What is the relationship between the students’ way of using the feedback that GeoGebra generates and the students´ reasoning?
- How do the students’ ways of reasoning and utilization of feedback from GeoGebra relate to their success in problem solving?

To examine the students’ reasoning and the utilization of feedback generated by GeoGebra, a didactical design (which will be presented in detail later) used in a previous study (Granberg & Olsson, submitted) was adopted. It was designed in line with didactical propositions of Brousseau (1997) and Schoenfeld (1985). Student dialogues, gestures, and screen activates were recorded and used as data.

FRAMEWORK AND BACKGROUND

The main components of the research questions are reasoning, feedback, and success of problem solving. The research questions concern the relationships between those components. To structure data Schoenfeld’s (1985) framework for protocol analysis was used. To answer the research questions, concepts of Lithner’s (2008) framework was used to analyze reasoning and concepts of Shute (2008) were used to analyze feedback. The paper ends in a discussion of the results from the perspective of the students´ attitudes to mathematics referring to Di Martino and Zan (2010). Each part of the framework and background will be further presented in the following paragraphs.

Problem solving

Schoenfeld (1985) elaborated and extended Pólya’s (1945) four problem-solving phases to the following six: Reading the task, analysis (why the properties of a task have certain consequences), exploration (why some outcomes will be useful), planning (why a certain approach would lead to solution), implementation (why the search for a
solution proceeds in a proper way) and verification (why a solution is actually reached). Focusing on decision-making at the executive or control level, Schoenfeld (1985) proposed a method of protocol analysis to examine the way decisions shape the path through problem solving. The protocol is based on the six phases of problem solving and the transitions between these phases. Protocols are parsed into episodes, which are periods of time during which the problem solvers are engaged in a single set of action of the same type or character such as planning, exploration, implementation, etc. Three classes of potential decision points are described; the junction between episodes, when new information or possibilities to take a new approach occur to the student, and when difficulties indicate that consideration of a change in approach is needed. In the present study Schoenfeld’s framework will be used in order to structure the students’ attempted solutions into episodes, and to consider whether certain decisions may be related to successfully finding a solution. The conversations and computer activities associated with determining a successful solution will be further analyzed through Lithner’s (2008) framework of reasoning.

**Reasoning**

Students who are attempting to solve mathematical tasks will engage in reasoning. Lithner (2008) defines the learner’s reasoning as her line of thought, that is, the thinking process during which the learner successfully or unsuccessfully attempts to solve a mathematical task. Reasoning is guided, as well as limited, by the student’s competences and is created in a sociocultural milieu. Lithner characterizes reasoning as imitative or creative.

During their quest for a solution to a given task, the students’ strategy may be to recall known facts, algorithms, or procedures that can be followed in order to reach an answer. Lithner (2008) associates these strategies to imitative reasoning, IR. One variant of IR is memorized reasoning, to recall memorized facts or complete answers, e.g. a proof, a definition, or that 1 liter = 1000 cm$^3$, but most school mathematics tasks ask for some kind of calculation or other process and such tasks can often be solved by algorithmic reasoning, to apply provided or memorized procedures and algorithms. This is often efficient if the algorithm is remembered correctly and then only a careless mistake may prevent the student from reaching a correct answer. Imitative strategies are described as memorizing and recalling, and often lead to rote learning.

Creative mathematical founded reasoning (CMR), is characterized by novelty, plausible argumentation and mathematical foundation. That is, instead of recalling a procedure that will solve the task, the students’ create methods that, at least to some extent, are new to them. The solution strategies may be supported by plausible argumentation, which is anchored in intrinsic mathematical properties of the involved mathematical components. Lithner (2008) suggests a wide conception of mathematical reasoning. In contrast to strict mathematical reasoning, which means distinguishing a guess from a proof, plausible reasoning includes also distinguishing a guess from a more reasonable guess. Plausible reasoning is not necessarily strictly logical but
constructive through the support of plausible arguments. The more plausible they are, the stronger the logical value.

In order to address the question of what an argument is, Lithner (2008) introduced the notion “anchoring”, which refers to its fastening in relevant mathematical properties of the components about which one is reasoning; objects, transformations, and concepts. The object is the mathematical component, the transformation is what one is doing with the object (a sequence of transformations is a procedure), and the outcome is another object. A concept is a mathematical idea that builds on objects, their transformations, and their properties. Depending on the purpose of a transformation, a mathematical property may be superficial or intrinsic. Lithner (2008) illustrates this in the following example (p.261): *In deciding if 9/15 or 2/3 is largest, the size of the numbers (9, 15, 2, 3) are a surface property that is insufficient to consider while the quotient captures the intrinsic property.* If the student, instead of applying a memorized procedure creates an original solution method (provided it’s not done by pure guesswork) it is necessary to construct arguments for why the method will solve the task. Argumentation may be considered as predictive or verificative. Relating to Schoenfeld’s problem solving phases presented above, in the phases of analysis, exploration, and planning, the arguments are primarily predictive. The phases’ implementation and verification include primarily verificative argumentation (Lithner, 2008).

**Feedback**

The students’ activities in GeoGebra may have a more or less articulated purpose of determining something particular. The actual computer activity, when the student’s input is entered and the result of GeoGebra’s processing appears on the screen, generates feedback associated to the action. In this study it is assumed that the computer activity is intended to contribute to the solving of the task and the information from GeoGebra is feedback. It is also assumed the student will use the feedback in different ways, e.g. to determine if they are right or wrong, to find clues how to proceed, etc.

According to Shute (2008), information meant as feedback to a learner in response to some action on the learner’s part can be delivered in different ways, e.g. verification of response accuracy, explanation of a correct answer, hints, worked examples, and can be administered at various occasions during or after the learning process. Feedback directed to the student’s activity is considered as having an effect on student learning. This is known as *Formative Feedback* and has the purpose of promoting learning (Shute, 2008). Shute’s definition of formative feedback is *information communicated to the learner that is intended to modify her thinking or behavior for the purpose of improving learning* (p.154). In this study the feedback from GeoGebra is a result of an activity planned by the students. The student may have an idea of what feedback she needs, she will shape the computer activity in relation to that purpose and may have the opportunity to use feedback from the software to modify and improve her learning.
Olsson

Shute (2008) found that a specific form of formative feedback, *Feedback on task-level*, is particularly effective for supporting learning. Compared to general summary feedback, feedback at the task level is more specific and often provides real-time information about a particular response to a problem or task to the student. In this study the feedback is considered as at the task-level.

Formative feedback consists of two parts affecting each other. In learning situations a teacher may give a response that is dependent on a student’s behavior, which in turn may affect the student’s behavior. Brousseau argues that feedback does not necessarily comes from a teacher or a peer; it may be a result of the student acting on the learning situation, which in turn will change as a result of the action. If the learning situation changes the student has to reconsider her behavior (Brousseau, 1997). In the current study, one of the main parts of the learning situation is the interface of GeoGebra. The dynamic software will respond according to the student’s activity and in turn affect the students’ actions. This will be considered as an example using feedback from the interactive software.

Formative Feedback provides students with two types of information: *Verification* and *Elaboration*. Verification involves confirming whether an answer is correct or incorrect and can be accomplished in different ways; explicit, e.g. a prepared piece of information from a teacher or, implicit, e.g. expected or unexpected results in a simulation. Elaboration has several variations, e.g. to address the response, discuss particular errors, provide worked examples or give gentle guidance. One type of elaboration, response specific feedback is considered as particularly learning-efficient. Response specific feedback focuses on the learners answer and may describe why or why not an answer is correct (Shute, 2008). In this study the feedback from software is considered as implicit and both verificative and elaborative. If the students have articulated a prediction of the outcome of an activity and just note whether the prediction is fulfilled or not, it is defined as verification. If the students discuss the outcome in terms why the results corresponded, or failed to correspond, with the predicted results or if the outcome is elaborated in some other way (above just noting if a prediction has been fulfilled) it is considered as elaboration on the feedback.

Students’ attitudes

Di Martino and Zan (2010) suggest instead of just explaining mathematics difficulties with negative attitudes, teachers should identify a student’s attitude profile in particular negative attitudes associated to mathematic difficulties. Instead of stating the difficulties depend on attitudes, teachers could act towards a change. Di Martino and Zan (2010) investigated a large number of student narratives about mathematics and found that attitudes often change over time and that this change is due to various situations.
METHOD

The method was designed to answer the research questions about reasoning, feedback, success of solving the task, and the relationship between those components. To collect data, a didactic situation, previously used by Granberg & Olsson (submitted), was adopted. In that study the didactic situation was found to engage students in reasoning and to use feedback generated by GeoGebra.

The didactic situation

The didactic situation was built on three propositions: challenge, responsibility, and collaboration. Schoenfeld (1985) argues that students need to work with mathematical problems that to some extent are new to them in order to develop problem solving skills. Brousseau (1997) proposed that if a task shall remain a challenge the students must have the responsibility to create their own methods for arriving at a solution. Furthermore Brousseau suggests that the teacher should instruct students until they can continue on their own, and then devolving the responsibility for solving the task to the students. During student-active sessions the teacher should not guide the students to right answers. If a task has an appropriate design, the students will reach the target knowledge for the task if they solve it. If the teacher offers information regarding how to solve the task, the students will not reach the knowledge target.

Working in small groups has been reported beneficial for learning under the circumstances that the task is focused on relationships and concepts rather than procedures. The former invites to collaboration and the latter to co-operation (Lou, Abrami, & d’Apollonia, 2001; Mullins, Rummel, & Spada, 2011). Collaboration is understood as a coordinated activity that is the result of a continued attempt to construct and maintain a shared conception of a problem (J. Roschelle & Teasley, 1994). In contrast, co-operation means that the co-operators split the task into parts and each one works with different parts.

The students worked in pairs sharing one computer using the software GeoGebra. The task consisted of creating three pairs of linear functions whose graphical representations were perpendicular and to formulate a rule for the circumstances when the graphs of two linear functions are perpendicular.

Sample and procedure

Sixteen students from the science program at a Swedish upper secondary school volunteered. They were 16 – 17 years old, 8 girls and 8 boys. They had some earlier experiences with linear functions but they had no recent instruction regarding the issue.

The students solved the task outside the classroom in pairs. They used a prepared GeoGebra-file, which contained a textbox with the instructions for the task and all tools were disabled except for the pointer, the “layer-mover”, and the angle-tool. They had a short introduction to GeoGebra, how to submit formulas into the input-field, how to change an algebraic expression and how to use the visible tools. Furthermore they were informed that they could ask for support on technical matters (how to handle
GeoGebra). In situations where students did not know how to proceed, the author encouraged them to explain what they had done and why they thought that their strategies did not work. When students felt they had solved the task they were asked why they were convinced they had come to a solution. Data was captured through screen recording, with integrated voice and video recording.

**Analysis method**

Research question 1 concerns the relationship between the students' reasoning and the feedback generated by GeoGebra. The students' reasoning was categorized using Lithner’s framework of creative and imitative reasoning (2008). The way that students used GeoGebra’s feedback was examined using the concepts verificative and elaborative feedback (Shute, 2008). The relationship between the students’ reasoning and GeoGebra’s feedback was analyzed by considering whether the students’ way of reasoning before and after a computer activity could be related to their way of using the feedback from GeoGebra. Research Question 2 concerns how the results from RQ1 relate to students' problem solving success. This will be analyzed by considering whether important decisions are consequences of certain reasoning and the use of feedback from GeoGebra. The analysis methods indicated here will be elaborated in the following text.

The data consisting of conversations, computer interactions, and gestures was transcribed into written text. In order to discuss the students' reasoning and their way of using feedback from GeoGebra in relation to their success in problem solving the eight pairs were divided into two groups; those who reached a reasonable solution and those who did not.

Schoenfeld’s protocol-analysis provides a way to examine the way students’ decisions shaped the way that solutions evolved (Schoenfeld, 1985, p.292). In order to structure data the transcripts were partitioned into episodes according to Schoenfeld’s six phases of problem solving, i.e. reading, analyzing, planning implementing, exploring, and verification. Thereafter possible decision points were identified, i.e. junctions between episodes, occasions where new information arose from computer activities or students’ discussions, and sequences accompanied by difficulties. Actual decisions, when students’ utterances or activities indicate how to proceed were noted. These parts were used to consider in what way the decisions contributed to solving parts of the task and if information gained from solving parts of the task were used to answer the main question of the task.

Lithner’s (2008) framework was used to classify students’ reasoning into IR or CMR. The students’ conversations, interaction with GeoGebra and gestures were examined and units of argumentation were identified. The characteristics of the argumentation, i.e. the implicit or explicit justifications of the strategy choices and the strategy implementations, were used to determine if the reasoning fulfilled the characterizations of imitative or creative reasoning (Lithner, 2008). The students’ reasoning was regarded as CMR if there were signs of creating a (for the students) new
method of solving the problem and if their argumentation was anchored in intrinsic mathematical concepts. The reasoning was categorized as imitative reasoning if the (sub) task solutions were based on familiar facts and/or procedures.

Finally, the way students used GeoGebra’s feedback was examined using the concepts verificative and elaborative feedback (Shute, 2008). Dialogues and gestures before and after each computer activity were noted. A computer activity in this study includes the student input and the outcome displayed by GeoGebra. Before this moment the students will plan (planning phase) what to submit to the software and afterwards the students may interpret the outcome and discuss how to proceed (verificative and analytic phase). An utterance in a planning phase when the students predicted the outcome of a computer activity was interpreted as preparation for using the information from GeoGebra as verifying feedback. After a computer activity, in the verificative phase, students could use the feedback from GeoGebra verifiably, identified as utterances of success or failure. If they, after the verification, used the information to explain, plan how to proceed with the task solving, etc. they were considered to be, using the information from the software elaborately, entering the analytic phase. Finally, the situations of preparing activities and using feedback from GeoGebra were compared to whether the reasoning were classified as either CMR or IR.

To answer RQ1, the use of feedback, verifiably and/or elaborately was associated to the characteristics of reasoning, IR or CMR during the planning of the activity, and to the reasoning when using feedback.

To answer RQ2 the reason for students’ success or failure in solving the task were related to decisions that the students made and could have made. It was then considered whether the success or failure was related to the characteristics of reasoning and to the use of feedback.

RESULTS

Six of eight pairs came to a reasonable solution for the main task. They used possible decision points for solving sub problems, and used gained information to solve the following sub problems and the main task. Two pairs did not reach a reasonable solution to the main task. Two pairs started out as the less successful pairs but changed strategy and completed the task as the more successful pairs. Since none of pairs had a clear understanding of the formula $y=mx+c$ they all needed to clarify the properties of the formula. The following example is from such a sequence and will exemplify analysis of reasoning and utilization of feedback.

Episodes and decision points

During their attempt at finding a solution, S1 and S2 went through episodes of reading, exploring, planning/implementation, analyzing, and verifying. They had possible decision points at the junctions of episodes and when the computer activity generated new information. Two of those decision points particularly supported their problem
solving. The first of these decision points emerged when they realized that they did not fully understand the formula \( y = mx + c \), and they decided to analyze the properties of the formula. The second decision point arose when they had difficulties locating a perpendicular function to \( y = 7x - 1 \), and they decided to change the function to \( y = 2x - 1 \) since (2) is easier to divide than (7). It was also clear that they used information from these episodes of analysis later in the task solving process. In the next paragraph their first episodes of exploring will be analyzed.

**Reasoning**

After reading the instructions they initiated an exploring episode as follows:

1. S2: well let’s just submit something...
2. S1: y is equal to seven....
3. S2: That means it’s going to be very much like this (pointing almost vertically, in front of the screen)

The suggestion to choose seven as the \( x \)-coefficient is followed by a prediction of the graphical appearance on the screen. They created the strategy themselves and S2’s utterance and gesture is interpreted as predictive argumentation. This strategy of suggesting something followed by a prediction of the result supported by argumentation, reappeared several times during their work. Some predictions were followed up by verificative argumentation, e.g. “\( m = 7 \) means the line must increase by 7 every step to the right”, or “this one must have \( m \) less than 1 because you go more steps horizontally than vertically”. Their reasoning is classified as CMR.

**Feedback**

The following excerpt, considered as an episode of analysis, will exemplify the way S1 and S2 used the information after submitting the function \( y = -3x - 1 \), which they predicted to have “negative but less slope than \( y = 7x - 1 \)”:

1. S1: This is not 90°....
2. S2: No it’s not... but let’s measure it to see how far off we are [uses GeoGebra's angle tool to measure the angle]...

After a discussion resulting in the conclusion that the constant term does not affect the slope of the function and that the slope depends only on \( m \), the \( x \)-coefficient:

3. S1: we must concentrate on \( m \)....

After an analysis of different examples of submitted functions S1 summarized using \( y = 2x - 1 \) and \( y = 7x - 1 \) as references:

1. S1: Well, if we start at minus one.... This one has \( m = 2 \).... Then you go one step to the right and then two upwards [counting squares with the mouse].... And this has \( m = 7 \).... if you go one step to the right you go seven upwards [counting squares with the mouse]....
First they used the GeoGebra’s feedback for verification, concluding that they did not have a perpendicular line, and then they initiated an attempt to elaborate on the result. This led them to an episode of analysis where they elaborated on the feedback and investigated the way m and c affect the graphical representation. During their work these students frequently discussed and elaborated on the feedback they received according to which they adjusted their strategies. This indicates that they were using feedback from the software both as verificative and elaborative feedback.

**Relations between reasoning, feedback, and success in problem solving**

S1 and S2 frequently used CMR to predict the outcome of the computer activities, and they used the feedback from GeoGebra both for verification and elaboration. Furthermore, these students always related their elaborations to their predictions. This indicates relationships between CMR and elaboration on feedback from GeoGebra. It seems that predictions of computer activities that are founded in CMR gives ground for elaborately using the feedback that is received.

S1’s and S2’s decisions to examine the formula $y=mx+c$ and to replace the $x$-coefficient of (7) with (2) are considered as important for solving the task. Both decisions were taken in episodes of analysis and preceded by elaboration on feedback based on CMR. Information from analysis was then used to answer the main question of the task. These students’ engagement in CMR, and their elaborative use of feedback in the episodes of analysis seems important for their success in solving the task. Examples of students that are not reasoning creatively and not using feedback elaborately are provided in the following paragraph.

Examples of imitative reasoning and only verifying utilization of feedback

This extract is from two students as is considered as exploring the circumstances for the task:

1. S3: c was where it intersects the y-axis…
2. S4: yes…
3. S3: yes it was… but what is m…
4. S4: m was that value in between…
5. S3: yes… the difference when you go…
6. S4: yes…
7. S3: eh… what should I write then…

The utterance on line 1 and 3, and the attempts to explain on line 4 and 5 indicate that these students are trying to remember the way c affects the intersection to the $y$-axis and the way to calculate the $x$-coefficient. The articulated facts are not coherent and there is no argumentation. This is characteristic for imitative reasoning. In the next excerpt there is an example of using information only verifying. The students have
submitted the function $y=-2x-2$ for the purpose of creating a perpendicular line to $y=2x-2$:

1. S3: no… that is not perpendicular… it is too large… submit $y=-1x-2$
2. S4: [submits $y=-1x-2$]… this is not 90°…
3. S3: no… but we can change this one (pointing at $y=2x-2$) into $y=1x-2$…

Instead of analyzing why $y=-2x-2$ did not appear perpendicular to $y=2x-2$ they changed the values of the two functions until they had perpendicular lines. They used the information as verifying feedback indicating if they were wrong or right but they did not elaborate on the feedback.

**DISCUSSION**

The results in this study are illuminated by the perspective of imitative and creative reasoning as well as the students’ way of using information from the software as feedback. Still there are unanswered questions about why they reason and use information as feedback in certain ways. Some of the explanations may be in the way the task is designed and some in the way GeoGebra invites interaction. What is not investigated in the study is the students’ attitudes towards mathematics and mathematics education. Anyhow, more or less implicitly there is an assumption in the reports of Hiebert and Grouws (2007), Lithner (2008), and Schoenfeld (1985) that students are guided into the attitude that mathematics education is about a teacher telling one how to solve certain task and that one is supposed to remember these methods and procedures and recall them when needed. The approach of student S3 and S4 might be due to teaching that essentially consisted of instructions and tasks that were prepared to solve as per the instructions, as well as student S1 and S2 may have experienced problem-oriented teaching. On the other hand Di Martino and Zan (2010) found that student attitudes are not stable and often change over time due to age, environment, topics, and activities. Di Martino and Zan suggest teachers should identify the students’ “attitude-profile”, in particular the significance for negative attitude associated to mathematics difficulties. Then the teachers can act purposefully to change the attitude of the student. It is paramount to make students aware of the importance of attitude, giving good examples, and identify success. Di Martino and Zan also point out the importance of adapting teaching towards activities focusing on processes rather than short questions and short answers. In summary, it is reasonable to assume that some of the differences regarding the students’ success solving the task in the study can be attributed to their attitudes toward mathematics, as well as the fact that this activity is suitable to work with students’ attitudes as well as their mathematical conceptualization.

**References**


